Get Ready for the Lesson

Read the introduction to Lesson 9-1 in your textbook.

How many rounds of play would be needed for a tournament with 100 players? 7

Read the Lesson

1. Indicate whether each of the following statements about the exponential function \( y = 10^x \) is true or false.

   a. The domain is the set of all positive real numbers. false  
   b. The y-intercept is 1. true  
   c. The function is one-to-one. true  
   d. The y-axis is an asymptote of the graph. false  
   e. The range is the set of all real numbers. false

2. Determine whether each function represents exponential growth or decay.

   a. \( y = 0.2(3)^x \) growth  
   b. \( y = 3(\frac{2}{3})^x \) decay  
   c. \( y = 0.4101^x \) growth

3. Supply the reason for each step in the following solution of an exponential equation.

   \[ 9^x - 1 = 27x \]
   \[ (3)^{2x} - 1 = 3^3x \]
   \[ 2(2x - 1) = 3x \]
   \[ 4x - 2 = 3x \]
   \[ x - 2 = 0 \]
   \[ x = 2 \]

   Original equation  
   Rewrite each side with a base of 3.  
   Power of a Power  
   Property of Equality for Exponential Functions  
   Distributive Property  
   Subtract 3x from each side.  
   Add 2 to each side.

Remember What You Learned

4. One way to remember that polynomial functions and exponential functions are different is to contrast the polynomial function \( y = x^2 \) and the exponential function \( y = 2^x \). Tell at least three ways they are different.

   Sample answer: In \( y = x^2 \), the variable \( x \) is a base, but in \( y = 2^x \), the variable \( x \) is an exponent. The graph of \( y = x^2 \) is symmetric with respect to the \( y \)-axis, but the graph of \( y = 2^x \) is not. The graph of \( y = x^2 \) touches the \( x \)-axis at \( (0, 0) \), but the graph of \( y = 2^x \) has the \( x \)-axis as an asymptote. You can compute the value of \( y = x^2 \) mentally for \( x = 100 \), but you cannot compute the value of \( y = 2^x \) mentally for \( x = 100 \).
Example 1
Sketch the graph of \(y = 0.1(4)^x\). Then state the function's domain and range.

The domain is all real numbers, while the range is the set of all positive real numbers.

Example 2
Determine whether each function represents exponential growth, decay, or neither.

\[
\begin{align*}
\text{a. } y &= 0.5(2)^x \\
\text{b. } y &= -2.3(2)^x \\
\text{c. } y &= 1.10(5)^x
\end{align*}
\]

Exercises
Sketch the graph of each function. Then state the function's domain and range.

\[
\begin{align*}
\text{1. } y &= 3(2)^x \\
\text{2. } y &= -2\left(\frac{3}{4}\right)^x \\
\text{3. } y &= 0.25(5)^x
\end{align*}
\]

Determine whether each function represents exponential growth, decay, or neither.

\[
\begin{align*}
\text{4. } y &= 0.3(1.2)^x \\
\text{5. } y &= -5\left(\frac{1}{3}\right)^x \\
\text{6. } y &= 3(10)^{-x}
\end{align*}
\]
9-1  **Practice**  
**Exponential Functions**

Sketch the graph of each function. Then state the function’s domain and range.

1. \(y = 3(2)^x\)
   - Domain: all real numbers; range: all positive numbers

2. \(y = 2^{\frac{3}{2}}\)
   - Domain: all real numbers; range: all positive numbers

Determine whether each function represents exponential growth or decay.

3. \(y = 3(6)^x\)  
   - Growth

4. \(y = \frac{3}{2}(\frac{2}{3})^x\)  
   - Decay

5. \(y = 10^{-x}\)  
   - Decay

6. \(y = 2(2.5)^x\)  
   - Growth

Write an exponential function whose graph passes through the given points.

7. \((0, 1)\) and \((-1, 3)\)
   - \(y = \left(\frac{1}{3}\right)^x\)

8. \((0, 4)\) and \((1, 12)\)
   - \(y = 4(3)^x\)

9. \((0, 3)\) and \((-1, 6)\)
   - \(y = \left(\frac{1}{2}\right)^x\)

10. \((0, 0.5)\) and \((1, 3)\)
    - \(y = 5(3)^x\)

Simplify each expression.

11. \((0.1)^{0.1}\) and \((1, 0.5)\)
    - \(y = 0.1(5)^x\)

12. \((0, 0.2)\) and \((1, 1.6)\)
    - \(y = 0.2(8)^x\)

Simplify each expression.

13. \(\sqrt[3]{3\sqrt[3]{3}}\)
    - \(27\)

14. \((\sqrt[6]{2})\sqrt[6]{3}\) \(x\sqrt[6]{4}\)
    - \(14\)

15. \(5\sqrt[3]{3} \cdot 5\sqrt[3]{3}\)
    - \(5\sqrt[3]{3}\)

16. \(3^n - n^3\)
    - \(x^2\pi\)

Solve each equation or inequality. Check your solution.

17. \(3^x > 9\) \(x > 2\)
    - \(18\)

18. \(2^x + 3 = 32\)
    - \(1\)

19. \(4^x = \frac{1}{4}\) \(x = -\frac{1}{2}\)
    - \(20\)

20. \(2^x - 2 = 16\)
    - \(4\)

21. \(3^x + 5 = 27^x\)
    - \(5\)

22. \(27^x = 3^{2x + 3}\)
    - \(3\)

23. \(2^x + 1 = 27^x + 4\)
    - \(14\)

24. \(3^x - 1 = 3^{2x - 1}\)
    - \(3\)

25. \(y = 12,000(2)^x\)  
   - Write an exponential function to model the population \(y\) of bacteria after \(x\) days.

26. How many bacteria are there after 6 days? \(768,000\)

27. **EDUCATION**  
   A college with a graduating class of 4000 students in the year 2005 predicts that it will have a graduating class of 4462 in 4 years. Write an exponential function to model the number of students \(y\) in the graduating class \(t\) years after 2005. \(y = 4000(1.05)^t\)
1. **Golf Balls** A golf ball manufacturer packs 3 golf balls into a single package. Three of these packages make a gift box. Three gift boxes make a value pack. The display shelf is high enough to stack 3 value packs one on top of the other. Three such columns of value packs make up a display front. Three display fronts can be packed in a single shipping box and shipped to various retail stores. How many golf balls are in a single shipping box?

2. **Folding** Kay folds a piece of paper in half over and over until it is at least 25 layers thick. How many times does she fold the paper in half and how many layers are there?

3. **Subscriptions** Subscriptions to an online arts and crafts club have been increasing by 20% every year. The club began with 40 members. Make a graph of the number of subscribers over the first 5 years of the club’s existence.

4. **Tennis Shoes** The cost of a pair of tennis shoes increases about 5.1% every year. About how much would a $50 pair of tennis shoes cost 25 years from now?

5. **Money** For Exercises 5–7, use the following information.

   Sam opened a savings account that accrues compound interest at a rate of 3% annually. Let \( P \) be the initial amount Sam deposited and let \( t \) be the number of years the account has been open.

   The amount of money in the account after \( t \) years is given by the equation:

   \[ A = P(1.03)^t. \]

6. If Sam opened the account with $500 and made no deposits or withdrawals, how much is in the account 10 years later?

   $671.96

7. What is the least number of years it would take for such an account to double in value?

   24 years

8. **Enrichment** Use these graphs for the problems below.

   The Family \( y = x^n \)

   The Family \( y = e^{mx} \)

1. Use the graph on the left to describe the relationship among the curves \( y = x^{1/2}, y = x^1, \) and \( y = x^2 \). For \( n = 1/2 \) and \( n = 2, \) the graphs are reflections of one another in the line with equation \( y = x^1 \).

2. Graph \( y = x^n \) for \( n = 3, 1/2, 1, -1, \) and 10 on the grid with \( y = x^{1/2}, y = x^3, \) and \( y = x^2 \).

   See students' graphs.

3. Which two regions in the first quadrant contain no points of the graphs of the family for \( y = x^n? \)

   \((x, y)|x \geq 1 \text{ and } 0 < y \leq 1\) and \((x, y)|0 < x \leq 1 \text{ and } y \geq 1\)

4. On the right grid, graph the members of the family \( y = e^{mx} \) for which \( m = 1 \) and \( m = -1 \).

   See students' graphs.

5. Describe the relationship among these two curves and the \( y \)-axis.

   the graphs for \( m = 1 \) and \( m = -1 \) are reflections in the \( y \)-axis.

6. Graph \( y = e^{mx} \) for \( m = 0, 1, -1, 1/2, -1/2, 2, -2, \) and \( 4 \).

   See students' graphs.
A graphing calculator can be used to determine a regression equation that best fits a set of data. This activity requires tiles labeled on one side, and a container.

Collect the Data

**Step 1** Place the tiles on the desktop and count the total number. Record the total number. Then place the tiles in the container and gently shake.

**Step 2** Pour the tiles onto the desktop, remove all the tiles with a label showing, and set these aside. Count the remaining tiles without the labels showing and return them to the container.

**Step 3** Record the data in a table like this one.

<table>
<thead>
<tr>
<th>Trials</th>
<th>Number of tiles without label showing</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

**Step 4** Repeat step 2 and 3 until the number of tiles without labels is zero or the number remains constant.

**Step 5** Take the tiles that were set aside in step 2 and pour them out of the container onto the desktop. Remove the tiles without the label showing and count the tiles with the label showing. Repeat this process until all the tiles have been removed.

**Step 6** Record the data in a table like this one.

<table>
<thead>
<tr>
<th>Trials</th>
<th>Number of tiles with the label showing</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Analyze the Data

1. Enter trials in L1 and number of tiles without label showing in L2. Enter trials in L3 and number of tiles with the label showing in L4.

2. Use [STATPLOT] to make a scatter plot. Make a graph on paper for each plot. Record the window used. Describe the pattern of the points.

3. From the STAT [CALC] menu find the regression equation that best fits the data. Record the two closest equations, rounding values to the nearest hundredths. List and discuss the r and/or r^2 values. Also include the graphs in determining the best-fitting regression equation.

4. Sketch your best-fit regression equation choice for each scatter-plot on paper.

5. Describe any problems with the data or the regression equations.

6. Insert (0, total number of tiles) in the tables and the lists. Describe the effect on the graphs. What happens with [PwrReg] and [ExpReg] when this ordered pair is inserted? Explain why this occurs.

3. Indicate whether each of the following statements about the exponential function y = log_b x is true or false.
   a. The y-axis is an asymptote of the graph. **true**
   b. The domain is the set of all real numbers. **false**
   c. The graph contains the point (5, 0). **false**
   d. The range is the set of all real numbers. **true**
   e. The y-intercept is 1. **false**

Remember What You Learned

4. An important skill needed for working with logarithms is changing an equation between logarithmic and exponential forms. Using the words base, exponent, and logarithm, describe an easy way to remember and apply the part of the definition of logarithm that says, “log_b x = y if and only if b^y = x.” **Sample answer:** In these equations, b stands for base. In log form, b is the subscript, and in exponential form, b is the number that is raised to a power. A logarithm is an exponent, so y, which is the log in the first equation, becomes the exponent in the second equation.
9-2 Study Guide and Intervention
Logarithms and Logarithmic Functions

Logarithmic Functions and Expressions

Definition of Logarithm with Base \( b \)
Let \( b \) and \( x \) be positive numbers, \( b \neq 1 \). The logarithm of \( x \) with base \( b \) is denoted \( \log_b x \) and is defined as the exponent \( y \) that makes the equation \( b^y = x \) true.

The inverse of the exponential function \( y = b^x \) is the logarithmic function \( x = \log_b y \). This function is usually written as \( y = \log_b x \).

Properties of Logarithmic Functions
1. The function is continuous and one-to-one.
2. The domain is the set of all real numbers.
3. The \( y \)-axis is an asymptote of the graph.
4. The range is the set of all real numbers.
5. The graph contains the point \((1, 0)\).

Example 1
Write an exponential equation equivalent to \( \log_4 243 = 5 \).

Example 2
Write a logarithmic equation equivalent to \( 6^{-3} = \frac{1}{216} \).

Example 3
Evaluate \( \log_3 16 \).

\( \sqrt[3]{16} = 2 \), so \( \log_3 16 = \frac{4}{3} \).

Exercises
Write each equation in logarithmic form.

1. \( 2^{7} = 128 \)
   \( \log_2 128 = 7 \)

2. \( 3^{-4} = \frac{1}{81} \)
   \( \log_3 \frac{1}{81} = -4 \)

3. \( \left(\frac{1}{3}\right)^{-2} = \frac{9}{4} \)
   \( \log_\frac{1}{3} \frac{9}{4} = 3 \)

Write each equation in exponential form.

4. \( \log_{15} 225 = 2 \)
   \( 15^2 = 225 \)

5. \( \log_{9} 81 = -3 \)
   \( 3^{-3} = \frac{1}{27} \)

6. \( \log_{25} 32 = \frac{5}{2} \)
   \( 4^{\frac{5}{2}} = 32 \)

Evaluate each expression.

7. \( \log_{9} 81 \)

8. \( \log_{6} 64 \)

9. \( \log_{30} 100,000 = 5.0 \)

10. \( \log_{2} 625 = 4 \)

11. \( \log_{8} 128 = -7 \)

12. \( \log_{10} 0.00001 = -5 \)

13. \( \log_{2} \frac{1}{128} = -7 \)

14. \( \log_{10} 0.00001 = -5 \)

15. \( \log_{10} \frac{1}{128} = -2.5 \)

Example 1
Solve \( \log_2 2x = 3 \).

Original equation
\( 2x = 2^3 \)

Simplify
\( x = 4 \)

The solution is \( x = 4 \).

Example 2
Solve \( \log_5 (4x - 3) < 3 \).

Original equation
\( 4x - 3 < 5^3 \)

Logarithmic to exponential inequality
\( 4x < 128 \)

Addition Property of Inequalities
\( 3 < x < 32 \)

Simplify
The solution is \( x \in \left(3, 32\right) \).

Exercises
Solve each equation or inequality.

1. \( \log_2 32 = 5 \)

2. \( \log_3 27 = 3 \)

3. \( \log_5 125 = 3 \)

4. \( \log_{10} 1000 = 3 \)

5. \( \log_{10} 100000 = 5 \)

6. \( \log_{10} 10 = 1 \)

7. \( \log_{10} 1 = 0 \)

8. \( \log_{10} \frac{1}{10} = -1 \)

9. \( \log_{10} x \) is undefined when \( x \leq 0 \)

10. \( \log_{10} x \) is undefined when \( x \leq 0 \)

11. \( \log_{10} x \) is undefined when \( x \leq 0 \)

12. \( \log_{10} x \) is undefined when \( x \leq 0 \)

13. \( \log_{10} x \) is undefined when \( x \leq 0 \)

14. \( \log_{10} x \) is undefined when \( x \leq 0 \)

15. \( \log_{10} x \) is undefined when \( x \leq 0 \)

16. \( \log_{10} x \) is undefined when \( x \leq 0 \)

17. \( \log_{10} x \) is undefined when \( x \leq 0 \)

18. \( \log_{10} x \) is undefined when \( x \leq 0 \)
Chapter 9

9-2 Practice
Logarithms and Logarithmic Functions

Write each equation in logarithmic form.
1. $2^3 = 8 \log_2 8 = 3$
2. $3^2 = 9 \log_3 9 = 2$

3. $3^2 = \frac{1}{8} \log_3 \frac{1}{8} = -2$
4. $\left(\frac{1}{3}\right)^3 = \frac{1}{9} \log_{\frac{1}{3}} \frac{1}{9} = 2$

Write each equation in exponential form.
5. $\log_3 243 = 5\log_3 3 = 5$
6. $\log_4 64 = 3 \log_4 4 = 3$

7. $\log_4 3 = \frac{1}{2} \log_2 3 = 3$
8. $\log_5 \left(\frac{1}{25}\right) = -2 \log_5 \frac{1}{25} = \frac{1}{25}$

Evaluate each expression.
9. $\log_5 25 = 2$
10. $\log_9 3 = \frac{1}{2}$

11. $\log_{10} 1000 = 3$
12. $\log_{25} 5 = \frac{1}{3}$

13. $\log_4 \frac{1}{64} = -3$
14. $\log_6 \frac{1}{625} = -4$

15. $\log_8 8^3 = 3$
16. $\log_5 \frac{1}{2} = \frac{1}{3}$

Solve each equation or inequality. Check your solutions.
17. $\log_3 x = 5 \ 243$
18. $\log_2 x = 3 \ 8$

19. $\log_4 y < 0 \ 0 < y < 1$
20. $\log_3 x = 3 \ 64$

21. $\log_2 n > -2 \ n > \frac{1}{4}$
22. $\log_3 6 = \frac{1}{2}$

23. $\log_5 (4x + 12) = 2 \ 6$
24. $\log_3 (4x - 4) > 5 \ x > 9$

25. $\log_3 (x + 2) = \log_3 (3x)$
26. $\log_5 (3y - 5) = \log_5 (2y + 3) \ y \geq 8$

SOUND
An equation for loudness, in decibels, is $L = 10 \log_{10} R$, where $R$ is the relative intensity of the sound. Sounds that reach levels of 120 decibels or more are painful to humans. What is the relative intensity of a sound? 1015

INVESTING
Maria invests $1000 in a savings account that pays 4% interest compounded annually. The value of the account $A$ at the end of five years can be determined from the equation $\log_2 1000 = \log_2 (1000)$. Find the value of $A$ to the nearest dollar. $1217$

Answers

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9-2 Word Problem Practice
Logarithms and Logarithmic Functions

1. **FISH** The population of silver carp has been growing in the Mississippi River. About every 3 years, the population doubles. Write logarithmic expression that gives the number of years it will take for the population to increase by a factor of ten.

\[3 \log_2 10\]

2. **POWERS** Haley tries to solve the equation \( \log_4 2 = x \). She got the wrong answer. What was her mistake? What should the correct answer be?

\[ \log_4 2 = 2 \]

3. **DIGITS** A computer programmer wants to write a formula that tells how many digits there are in a number \( n \), where \( n \) is a positive integer. For example, if \( n = 343 \), the formula should evaluate to 3 and if \( n = 10,000 \), the formula should evaluate to 5. Suppose \( 8 = \log_{10} n < 9 \). How many digits does \( n \) have?

\[ \text{9}\]

4. **LOGARITHMS** Pauline knows that \( \log_3 x = 3 \) and \( \log_5 y = 5 \). She knows that this is the same as knowing that \( b^3 = x \) and \( b^5 = y \). Multiply these two equations together and rewrite it as an equation involving logarithms. What is \( \log_3 xy \)?

\[ \log_3 xy \]

MUSIC For Exercises 5 and 6, use the following information.

The first note on a piano keyboard corresponds to a pitch with a frequency of 27.5 cycles per second. With every successive note you go up the white and black keys of a piano, the pitch multiplies by a factor of \( \sqrt[12]{2} \). The formula for the frequency of the pitch sounded when the \( n \)th note up the keyboard is played is given by

\[ f = 27.5(\sqrt[12]{2})^n \]

5. The pitch that orchestras tune to is the A above middle C. It has a frequency of 440 cycles per second. How many notes up the piano keyboard is this A?

\[ \text{49}\]

6. Another pitch on the keyboard has a frequency of 1760 cycles per second. How many notes up the keyboard will this be found?

\[ \text{73}\]

9-2 Enrichment
Musical Relationships

The frequencies of notes that are one octave apart in a musical scale are related by an exponential equation. For the eight C notes on a piano, the equation is \( C_n = C_1 2^{n-1} \), where \( C_n \) represents the frequency of note \( C_n \).

MusicAl Relationships

1. Find the relationship between \( C_1 \) and \( C_2 \). \( C_2 = 2C_1 \)

2. Find the relationship between \( C_1 \) and \( C_4 \). \( C_4 = 8C_1 \)

The frequencies of consecutive notes are related by a common ratio \( r \). The general equation is \( f_n = f_1 r^{n-1} \).

3. If the frequency of middle C is 261.6 cycles per second and the frequency of the next higher C is 523.2 cycles per second, find the common ratio \( r \). (Hint: The two Cs are 12 notes apart.) Write the answer as a radical expression.

\[ r = \sqrt[12]{2} \]

4. Substitute decimal values for \( r \) and \( f_1 \) to find a specific equation for \( f_n \).

\[ f_n = 261.1(1.05946)^{n-1} \]

5. Find the frequency of F above middle C.

\[ f_2 = 261.6(1.05946)^1 = 277.18 \]

6. Frets are a series of ridges placed across the fingerboard of a guitar. They are spaced so that the sound made by pressing a string against one fret has about 1.0595 times the wavelength of the sound made by using the next fret. The general equation is \( w_n = w_0 / (1.0595)^n \). Describe the arrangement of the frets on a guitar.

The frets are spaced in a logarithmic scale.
Lesson Reading Guide
Properties of Logarithms

Get Ready for the Lesson

Read the introduction to Lesson 9-3 in your textbook.

1. Find the value of each of the following.
   a. \( \log_2 125 \)  
   b. \( \log_5 1 \)  
   c. \( \log_5 (125 - 5) \)

2. Which of the following statements is true? B
   A. \( \log_5 (125 - 5) = (\log_5 125) + (\log_5 5) \)
   B. \( \log_5 (125 - 5) = \log_5 125 - \log_5 5 \)

Read the Lesson

1. Each of the properties of logarithms can be stated in words or in symbols. Complete the statements of these properties in words.
   a. The logarithm of a quotient is the ______ difference ______ of the logarithms of the ______ numerator ______ and the ______ denominator ______.
   b. The logarithm of a power is the ______ product ______ of the logarithm of the base and the ______ exponent ______.
   c. The logarithm of a product is the ______ sum ______ of the logarithms of its ______ factors ______.

2. State whether each of the following equations is true or false. If the statement is true, name the property of logarithms that is illustrated.
   a. \( \log_2 10 = \log_2 30 - \log_2 3 \) true; Quotient Property
   b. \( \log_5 12 = \log_5 4 + \log_5 3 \) false
   c. \( \log_5 81 = 2 \log_5 9 \) true; Power Property
   d. \( \log_5 30 - \log_5 5 - \log_5 6 \) false

3. The algebraic process of solving the equation \( \log_2 x + \log_2 (x + 2) = 3 \) leads to \( x = -4 \) or \( x = 2 \). Does this mean that both \( -4 \) and \( 2 \) are solutions of the logarithmic equation? Explain your reasoning. Sample answer: No; \( 2 \) is a solution because it checks: \( \log_2 2 + \log_2 (2 + 2) = \log_2 2 + \log_2 4 = 1 + 2 = 3 \). However, because \( \log_2 (-4) \) and \( \log_2 (-2) \) are undefined, \( -4 \) is an extraneous solution and must be eliminated. The only solution is 2.

Remember What You Learned

4. A good way to remember something is to relate it something you already know. Use words to explain how the Product Property for exponents can help you remember the Product Property for logarithms. Sample answer: When you multiply two numbers or expressions with the same base, you add the exponents and keep the same base. Logarithms are exponents, so to find the logarithm of a product, you add the logarithms of the factors, keeping the same base.
**Solve Logarithmic Equations** You can use the properties of logarithms to solve equations involving logarithms.

**Example** Solve each equation.

a. \(2 \log_3 x - \log_3 4 = \log_3 25\)

\[
2 \log_3 x - \log_3 4 = \log_3 25 \quad \text{Original equation}
\]

\[
\log_3 x^2 = \log_3 100 \quad \text{Power Property}
\]

\[
x^2 = 100 \quad \text{Property of Equality for Logarithmic Functions}
\]

\[
x = 10 \quad \text{Take the square root of each side.}
\]

Since logarithms are undefined for \(x < 0\), the only solution is 10.

b. \(\log_2 x + \log_2 (x + 2) = 3\)

\[
\log_2 x + \log_2 (x + 2) = 3 \quad \text{Original equation}
\]

\[
\log_2 x(x + 2) = 3 \quad \text{Product Property}
\]

\[
x^2 + 2x = 8 \quad \text{Definition of logarithm}
\]

\[
x^2 + 2x - 8 = 0 \quad \text{Subtract 8 from each side.}
\]

\[
(x + 4)(x - 2) = 0 \quad \text{Factor.}
\]

\[
x = 2 \quad \text{or} \quad x = -4 \quad \text{Zero Product Property}
\]

Since logarithms are undefined for \(x < 0\), the only solution is 2.

**Exercises**

Solve each equation. Check your solutions.

1. \(\log_2 4 + \log_2 2x = \log_2 24\) \(3\)

2. \(3 \log_4 6 - \log_4 8 = \log_4 x\) \(27\)

3. \(\frac{1}{2} \log_2 25 + \log_2 x = \log_2 20\) \(4\)

4. \(\log_4 4 - \log_4 (x + 3) = \log_2 8 - \frac{5}{2}\)

5. \(\log_2 2x - \log_2 3 = \log_2 (x - 1)\) \(3\)

6. \(2 \log_4 (x + 1) = \log_4 (11 - x)\) \(2\)

7. \(\log_2 x - 3 \log_2 5 = 2 \log_2 10\) \(12,500\)

8. \(3 \log_2 x - 2 \log_2 5x = 2\) \(100\)

9. \(\log_3 (c + 3) - \log_3 (4c - 1) = \log_3 5\) \(\frac{18}{19}\)

10. \(\log_3 (x + 3) - \log_3 (2x - 1) - 2\) \(\frac{4}{7}\)

11. \(\log_{10} 27 = 3 \log_{10} x\) \(3\)

12. \(3 \log_2 4 = 2 \log_2 8\) \(8\)

13. \(\log_4 5 + \log_4 x = \log_4 60\) \(12\)

14. \(\log_6 4x + \log_6 8 = \log_6 80\) \(5\)

15. \(\log_5 y - \log_5 3 = \log_5 7\) \(21\)

16. \(\log_2 q - \log_2 3 = \log_2 7\) \(21\)

17. \(\log_5 4 + 2 \log_5 5 = \log_5 w\) \(100\)

18. \(3 \log_3 2 - 2 \log_3 4 = \log_3 b\) \(2\)

19. \(\log_{10} x + \log_{10} (3x - 5) = \log_{10} 2\) \(2\)

20. \(\log_4 x + \log_4 (2x - 3) = \log_4 2\) \(2\)

21. \(\log_5 d + \log_5 3 = 3\) \(9\)

22. \(\log_{10} y - \log_{10} (2 - y) = 0\) \(1\)

23. \(\log_2 s + 2 \log_2 5 = 0\) \(\frac{1}{25}\)

24. \(\log_4 (x + 4) - \log_4 (x - 3) = 3\) \(4\)

25. \(\log_4 (n + 1) - \log_4 (n - 2) = 1\) \(3\)

26. \(\log_3 10 + \log_3 12 = 3 \log_3 2 + \log_3 15\) \(5\)
9-3 Practice
Properties of Logarithms
Use $\log_{10} 5 \approx 0.6990$ and $\log_{10} 7 \approx 0.8451$ to approximate the value of each expression.

1. $\log_{10} 35 = 1.5441$
2. $\log_{10} 25 = 1.3980$
3. $\log_{10} 7^5 = 3.0161$
4. $\log_{10} 5^{-7} = -0.1461$
5. $\log_{10} 245 = 2.3892$
6. $\log_{10} 175 = 2.2431$
7. $\log_{10} 0.2 \approx -0.6990$
8. $\log_{10} 25^3 = 0.5529$

Solve each equation, check your solutions.
9. $\log_{61} n = \frac{2}{3} \log_{61} 8$
10. $\log_{10} u = \frac{2}{3} \log_{10} 4$
11. $\log_{8} x + \log_{8} 9 = \log_{8} 54$
12. $\log_{10} 48 - \log_{10} 4 = \log_{10} 12$
13. $\log_{2} (3u + 14) - \log_{2} 5 + \log_{2} 2u = 2$
14. $4 \log_{2} x + \log_{2} 5 - \log_{2} 405 = 3$
15. $\log_{3} y = -\log_{3} 16 + \frac{1}{3} \log_{3} 64$
16. $\log_{2} d = 5 \log_{2} 4 - 2 \log_{2} 8$
17. $\log_{10} (3m - 5) = \log_{10} m = \log_{10} 2$
18. $\log_{10} (b + 3) + \log_{10} b = \log_{10} 4$
19. $\log_{10} (x^2 + 10) - \log_{10} (x - 1) = \log_{10} 12$
20. $\log_{2} (a + 3) + \log_{2} (a + 2) = \log_{2} 6$
21. $\log_{5} (r^4) - \log_{5} r = \log_{5} (r + 1)$
22. $\log_{2} (x^4 - 4) - \log_{2} (x + 2) = \log_{2} 4$
23. $\log_{3} 10 + \log_{3} w = 2$
24. $\log_{3} (n - 3) + \log_{3} (n + 4) = 1$
25. $3 \log_{6} (x^2 + 9) - 6 = 0$
26. $\log_{50} (9x + 5) - \log_{50} (c^2 - 1) = \frac{1}{2}$
27. $\log_{10} (2x - 5) + 1 = \log_{10} (7x + 10)$
28. $\log_{2} (5y + 2) - 1 = \log_{2} (1 - 2y)$
29. $\log_{10} (c^2 - 1) - 2 = \log_{10} (c + 1)$
30. $\log_{7} x + 2 \log_{7} x - \log_{7} 3 - \log_{7} 72 = 6$

31. SOUND Recall that the loudness $L$ of a sound in decibels is given by $L = 10 \log_{10} R$, where $R$ is the sound's relative intensity. If the intensity of a certain sound is tripled, by how many decibels does the sound increase? About 4.8 dB

32. EARTHQUAKES An earthquake rated at 3.5 on the Richter scale is felt by many people, and an earthquake rated at 4.5 may cause local damage. The Richter scale magnitude reading $m$ is given by $m = \log_{10} x$, where $x$ represents the amplitude of the seismic wave causing ground motion. How many times greater is the amplitude of an earthquake that measures 4.5 on the Richter scale than one that measures 3.5? 10 times

9-3 Word Problem Practice
Properties of Logarithms
1. MENTAL COMPUTATION Jessica has memorized $\log_{10} 2 \approx 0.3010$ and $\log_{10} 5 \approx 0.6990$. Using this information, to the nearest thousandth, what power of 5 is equal to 6? 0.322

2. POWERS A chemist is formulating an acid. The pH of a solution is given by $-\log_{10} C$, where $C$ is the concentration of hydrogen ions. If the concentration of hydrogen ions is increased by a factor of 100, what happens to the pH of the solution? The pH decreases by 2.

3. LUCKY MATH Frank is solving a problem involving logarithms. He does everything correctly except for one thing. He mistakenly writes $\log_{9} (3x - 1) = \log_{9} 25$. However, after substituting the values in his problem, he amazingly still gets the right answer! The value of $x$ was 11. What must the value of $x$ have been? 11

4. LENGTHS Charles has two poles. One pole has length equal to $\log_{7} 21$ and the other has length equal to $\log_{7} 25$. Express the length of both poles joined end to end as the logarithm of a single number. $\log_{7} 525$

5. Derive an expression for $S$ and apply it to a cube in terms of $c$ where $c$ is the side length of a cube. $\log_{3} f$

6. How many cubes, each one foot on a side, would have to be put together to get an object that Alicia would call "big"? 729

7. How likely is it that a large object attached to a big object would result in a huge object, according to Alicia's scale? Not very likely; most likely the result will be big, not huge.
Spirals

Consider an angle in standard position with its vertex at a point \( O \) called the pole. Its initial side is on a coordinatized axis called the polar axis. A point \( P \) on the terminal side of the angle is named by the polar coordinates \((r, \theta)\), where \( r \) is the directed distance of the point from \( O \) and \( \theta \) is the measure of the angle. Graphs in this system may be drawn on polar coordinate paper such as the kind shown below.

1. Use a calculator to complete the table for \( \log_2 \theta = \frac{\theta}{120} \).

\[
\begin{array}{c|cccccccc}
 r & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 \theta & 0^o & 120^o & 190^o & 240^o & 279^o & 310^o & 337^o & 360^o \\
\end{array}
\]

(Hint: To find \( \theta \) on a calculator, press 120 \( \div \) LOG \( r \) \( \div \) LOG 2 \( \div \).)

2. Plot the points found in Exercise 1 on the grid above and connect to form a smooth curve.

This type of spiral is called a logarithmic spiral because the angle measures are proportional to the logarithms of the radii.

Get Ready for the Lesson

Read the introduction to Lesson 9-4 in your textbook.

Which substance is more acidic, milk or tomatoes? tomatoes

Read the Lesson

1. Rhonda used the following keystrokes to enter an expression on her graphing calculator:

\[
\text{LOG} 17 \text{ ENTER}
\]

The calculator returned the result 1.230448921.

Which of the following conclusions are correct? a, c, and d

a. The base 10 logarithm of 17 is about 1.2304.
b. The base 17 logarithm of 10 is about 1.2304.
c. The common logarithm of 17 is about 1.230449.
d. \( 10^{1.230448921} \) is very close to 17.
e. The common logarithm of 17 is exactly 1.230448921.

2. Match each expression from the first column with an expression from the second column that has the same value.

| a. \( \log_2 2 \) | iv | i. \( \log_{10} 1 \) |
| b. \( \log_{12} 1 \) | iii | ii. \( \log_{20} 8 \) |
| c. \( \log_3 1 \) | i | iii. \( \log_{30} 12 \) |
| d. \( \log_{\frac{1}{5}} 5 \) | v | iv. \( \log_5 5 \) |
| e. \( \log_{1000} 2 \) | ii | v. \( \log_0 0.1 \) |

3. Calculators do not have keys for finding base 8 logarithms directly. However, you can use a calculator to find \( \log_8 20 \) if you apply the change of base formula. Which of the following expressions are equal to \( \log_8 20 \)? B and C

A. \( \log_{10} 8 \) B. \( \log_{10} 20 \) C. \( \log_{10} 8 \) D. \( \log_{20} 8 \)

Remember What You Learned

4. Sometimes it is easier to remember a formula if you can state it in words. State the change of base formula in words. Sample answer: To change the logarithm of a number from one base to another, divide the log of the original number in the old base by the log of the new base in the old base.
Common Logarithms

Base 10 logarithms are called common logarithms. The expression \( \log_{10} x \) is usually written without the subscript as \( \log x \). Use the LOG key on your calculator to evaluate common logarithms.

The relation between exponents and logarithms gives the following identity.

\[
\begin{align*}
\log_b a &= \frac{\log_c a}{\log_c b} \\
&= \frac{\log a}{\log b} \\
&= \log_{c^n} a
\end{align*}
\]

Example 1

Evaluate \( \log 50 \) to four decimal places.

Use the LOG key on your calculator. To four decimal places, \( \log 50 \approx 1.6990 \).

Example 2

Solve \( 3x + 1 = 12 \).

Simplify.

\[
\begin{align*}
2x &= 11 \\
x &= 5.5
\end{align*}
\]

Exercises

Use a calculator to evaluate each expression to four decimal places.

1. \( \log 18 \) 
   1.2553
2. \( \log 39 \) 
   1.5911
3. \( \log 120 \) 
   2.0792

4. \( \log 5.8 \) 
   0.7634
5. \( \log 42.3 \) 
   1.6263
6. \( \log 0.003 \) 
   -2.5229

Solve each equation or inequality. Round to four decimal places.

7. \( 4x = 12 \) 
   \( x = 3.0000 \)
8. \( 6x + 2 = 18 \) 
   \( x = 2.6667 \)

9. \( 5x - 2 = 120 \) 
   \( x = 25.2000 \)
10. \( 7x - 1 \geq 21 \) 
    \( x \geq 4 \)

11. \( 2x + 4 = 30 \) 
    \( x = 13 \)
12. \( 6.5x \geq 200 \) 
    \( x \geq 31 \)

13. \( 3.8x - 1 = 85.4 \) 
    \( x = 25 \)
14. \( 2x + 5 = 3x - 2 \) 
    \( x = 7 \)
15. \( 9x = 6x + 2 \) 
    \( x = 0.6667 \)
16. \( 6x - 5 = 2x + 3 \) 
    \( x = 4 \)

Answers
9-4 Practice
Common Logarithms

Use a calculator to evaluate each expression to four decimal places.

1. \( \log_{10} 0.7782 = 0.8754 \)
2. \( \log_{10} 15 = 1.1791 \)
3. \( \log_{10} 1.0414 = 0.0414 \)
4. \( \log_{10} 0.3 = -0.5229 \)

Use the formula \( pH = -\log[H^+] \) to find the \( pH \) of each substance given its concentration of hydrogen ions.

5. \( H^+ \) concentration of hydrogen ions: milk: \( [H^+] = 1.0 \times 10^{-5} \) mole per liter \( pH = 5.6 \)
6. \( H^+ \) concentration of hydrogen ions: acid rain: \( [H^+] = 1.0 \times 10^{-6} \) mole per liter \( pH = 7.6 \)
7. \( H^+ \) concentration of hydrogen ions: gastric juices: \( [H^+] = 1.0 \times 10^{-4} \) mole per liter \( pH = 4.1 \)
8. \( H^+ \) concentration of hydrogen ions: tomato juice: \( [H^+] = 1.0 \times 10^{-1} \) mole per liter \( pH = 1.0 \)
9. \( H^+ \) concentration of hydrogen ions: toothpaste: \( [H^+] = 1.0 \times 10^{-4} \) mole per liter \( pH = 2.5 \)

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places.

23. \( \log_{10} 7 = \frac{\log_{10} 7}{\log_{10} 10} = 0.8451 \)
24. \( \log_{10} 66 = \frac{\log_{10} 66}{\log_{10} 10} = 1.8403 \)
25. \( \log_{10} 35 = \frac{\log_{10} 35}{\log_{10} 10} = 1.5579 \)
26. \( \log_{10} 36 = \frac{\log_{10} 36}{\log_{10} 10} = 1.5579 \)

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places.

27. \( \log_{10} 2 = \frac{\log_{10} 2}{\log_{10} 10} = 0.3010 \)
28. \( \log_{10} 3 = \frac{\log_{10} 3}{\log_{10} 10} = 0.4771 \)
29. \( \log_{10} 5 = \frac{\log_{10} 5}{\log_{10} 10} = 0.6990 \)
30. \( \log_{10} 7 = \frac{\log_{10} 7}{\log_{10} 10} = 0.8451 \)

Solve each equation or inequality. Round to four decimal places.

31. \( 2^x = 12 \) \( x = 3.5623 \)
32. \( 3^y = 27 \) \( y = 3.0000 \)
33. \( 10^z = 100 \) \( z = 2.0000 \)
34. \( 5^w = 125 \) \( w = 3.0000 \)
35. \( 2^v = 4 \) \( v = 2.0000 \)
36. \( 3^t = 9 \) \( t = 2.0000 \)
37. \( 10^s = 1000 \) \( s = 3.0000 \)
38. \( 2^u = 8 \) \( u = 3.0000 \)
39. \( 3^p = 27 \) \( p = 3.0000 \)
40. \( 10^q = 10000 \) \( q = 4.0000 \)
41. \( 2^r = 4 \) \( r = 2.0000 \)
42. \( 3^s = 27 \) \( s = 3.0000 \)
43. \( 10^t = 100000 \) \( t = 5.0000 \)
44. \( 2^u = 8 \) \( u = 3.0000 \)
45. \( 3^v = 27 \) \( v = 3.0000 \)
46. \( 10^w = 1000000 \) \( w = 6.0000 \)
47. \( 2^x = 128 \) \( x = 7.0000 \)
48. \( 3^y = 243 \) \( y = 5.0000 \)
49. \( 10^z = 1000 \) \( z = 3.0000 \)
50. \( 2^t = 32 \) \( t = 5.0000 \)
51. \( 3^u = 27 \) \( u = 3.0000 \)
52. \( 10^v = 1000 \) \( v = 3.0000 \)
53. \( 2^w = 8 \) \( w = 3.0000 \)
54. \( 3^x = 27 \) \( x = 3.0000 \)
55. \( 10^y = 1000 \) \( y = 3.0000 \)
56. \( 2^z = 8 \) \( z = 3.0000 \)
57. \( 3^a = 27 \) \( a = 3.0000 \)
58. \( 10^b = 1000 \) \( b = 3.0000 \)
59. \( 2^c = 8 \) \( c = 3.0000 \)
60. \( 3^d = 27 \) \( d = 3.0000 \)
61. \( 10^e = 1000 \) \( e = 3.0000 \)
62. \( 2^f = 8 \) \( f = 3.0000 \)
63. \( 3^g = 27 \) \( g = 3.0000 \)
64. \( 10^h = 1000 \) \( h = 3.0000 \)
65. \( 2^i = 8 \) \( i = 3.0000 \)
66. \( 3^j = 27 \) \( j = 3.0000 \)
67. \( 10^k = 1000 \) \( k = 3.0000 \)
68. \( 2^l = 8 \) \( l = 3.0000 \)
69. \( 3^m = 27 \) \( m = 3.0000 \)
70. \( 10^n = 1000 \) \( n = 3.0000 \)
1. **OTHER BASES** Jamie needs to figure out what \( \log_2 3 \) is, but she only has a table of common logarithms. In the table, she finds that \( \log_{10} 2 = 0.3010 \) and \( \log_{10} 3 = 0.4771 \). Using this information, to the nearest thousandth, what is \( \log_2 3 \)?

**Answer:**

\[ \log_2 3 = \log_{10} 2 \cdot \log_{10} 3 = 0.3010 \cdot 0.4771 = 0.143 \]

2. **PH** The pH of a solution is given by

\[ \text{pH} = -\log_{10} C \]

where \( C \) is the concentration of hydrogen ions in moles per liter. A solution of baking soda creates a hydrogen ion concentration \( 5 \times 10^{-9} \) of mole per liter. What is the pH of a solution of baking soda? Round your answer to the nearest tenth.

**Answer:**

\[ \text{pH} = -\log_{10} (5 \times 10^{-9}) = 8.3 \]

3. **GRAPHING** The graph of \( y = \log_{10} x \) is shown below. Use the fact that

\[ \frac{1}{\log_{10} 2} = 3.32 \]

to sketch a graph of \( y = \log_2 x \) on the same graph.

![Graph of \( y = \log_{10} x \) and \( y = \log_2 x \)]

4. **SCIENTIFIC NOTATION** When a number \( n \) is written in scientific notation, it has the form \( n = s \times 10^p \), where \( s \) is a number greater than or equal to 1 and less than 10 and \( p \) is an integer. Show that \( n = \log_{10} C, \) \( n = \log_2 \sqrt{2} \), and \( n = \log_{10} 2 \).

**Answers:**

- \( \log_{10} C = n \) (where \( C \) is the concentration of hydrogen ions in moles per liter)
- \( \log_2 \sqrt{2} = \frac{1}{2} \log_2 2 = 0.5 \)
- \( \log_{10} 2 = 0.3010 \)

5. **LOG TABLE** For Exercises 5 and 6, use the following information.

Marjorie is looking through some old science books owned by her grandfather. At the back of one of them, there is a table of logarithms base 10. However, the book is worn out and some of the entries are unreadable.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \log_{10} x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.3010</td>
</tr>
<tr>
<td>3</td>
<td>0.4771</td>
</tr>
<tr>
<td>4</td>
<td>0.6020</td>
</tr>
<tr>
<td>5</td>
<td>0.6989</td>
</tr>
<tr>
<td>6</td>
<td>?</td>
</tr>
</tbody>
</table>

**Approximately what are the missing entries in the table?** Round off your answers to the nearest thousandth.

- \( \log_{10} 4 = 0.602 \)
- \( \log_{10} 6 = 0.778 \)

**Sample answer:**

\[ \log_{10} 1.5 = \log_{10} 1.5 = \log_{10} 2 + \log_{10} 3 - \log_{10} 10 \]

\[ \approx 0.1760 \]

6. **How can you use this table to determine \( \log_{10} 1.57 \)?**

**Sample answer:**

\[ \log_{10} 1.57 = \log_{10} 1.5 + \log_{10} 3 - \log_{10} 10 \]

\[ \approx 0.4771 + 0.4771 - 1 = 0.4742 \]

**The Slide Rule**

Before the invention of electronic calculators, computations were often performed on a slide rule. A slide rule is based on the idea of logarithms. It has two movable rods labeled with C and D scales. Each of the scales is logarithmic.

To multiply \( 2 \times 3 \) on a slide rule, move the C rod to the right as shown below. You can find \( 2 \times 3 \) by adding \( \log 2 \) to \( \log 3 \), and the slide rule adds the lengths for you. The distance you get is 0.778, or the logarithm of 6.

**Follow the steps to make a slide rule.**

1. Use graph paper that has small squares, such as 10 squares to the inch. Using the scales shown at the right, plot the curve \( y = \log x \) for \( x = 1, 1.5, \) and the whole numbers from 2 through 10. Make an obvious heavy dot for each point plotted.

2. You will need two strips of cardboard. A 5-by-7 index card, cut in half the long way, will work fine. Turn the graph you made in Exercise 1 sideways and use it to mark a logarithmic scale on each of the two strips. The figure shows the mark for 2 being drawn.

3. Explain how to use a slide rule to divide 8 by 2. Line up the 2 on the C scale with the 8 on the D scale. The quotient is the number on the D scale below the 1 on the C scale.
9-5 Lesson Reading Guide
Base e and Natural Logarithms

Get Ready for the Lesson

Read the introduction to Lesson 9-5 in your textbook.

Suppose that you deposit $675 in a savings account that pays an annual interest rate of 5%. In each case listed below, indicate which method of compounding would result in more money in your account at the end of one year.

a. annual compounding or monthly compounding
b. quarterly compounding or daily compounding
c. daily compounding or continuous compounding

Read the Lesson

Example 1
Jagdish entered the following keystrokes in his calculator:

\[ \text{LN } 5 \quad \text{LN} \quad \text{ENTER} \]

The calculator returned the result 1.609437912. Which of the following conclusions are correct?

a. The common logarithm of 5 is about 1.6094.
b. The natural logarithm of 5 is about 1.6094.
c. The base 5 logarithm of \( e \) is about 1.6094.
d. The natural logarithm of 5 is about 1.609438.
e. \( e^{1.609437912} \) is very close to 5.

Example 2
Evaluate ln 1685.

Example 3
Evaluate ln 1685.

Exercises

1. ln 34
2. ln 84350
3. ln 0.735
4. ln 100
5. ln 0.0824
6. ln 2.388
7. ln 128,245
8. ln 0.00614

Write an equivalent exponential or logarithmic equation.

9. \( e^x = 8 \)
10. \( e^{2x} = 45 \)
11. \( \ln 20 = x \)
12. \( \ln x = 8 \)

Evaluate each expression.

17. \( \ln e^3 \)
18. \( \ln 42 \)
19. \( \ln 0.5 \)
20. \( \ln e^{16/2} \)
Study Guide and Intervention

Base e and Natural Logarithms

Equations and Inequalities with e and ln

All properties of logarithms from earlier lessons can be used to solve equations and inequalities with natural logarithms.

Example

Solve each equation or inequality.

a. \(3e^{2x} + 2 = 10\)
   
   \[3e^{2x} + 2 = 10\] Original equation
   
   \[3e^{2x} = 8\] Subtract 2 from each side.
   
   \[e^{2x} = \frac{8}{3}\] Divide each side by 3.
   
   \[2x = \ln \frac{8}{3}\] Property of Equality for Logarithms
   
   \[x = \frac{1}{2} \ln \frac{8}{3}\] Multiply each side by \(\frac{1}{2}\).
   
   \[x \approx 0.4904\] Use a calculator.

b. \(\ln (4x - 1) < 2\)
   
   \[\ln (4x - 1) < 2\] Original inequality
   
   \[e^{\ln (4x - 1)} < e^2\] Write each side using exponents and base e.
   
   \[0 < 4x - 1 < e^2\] Inverse Property of Exponents and Logarithms
   
   \[1 < 4x < e^2 + 1\] Addition Property of Inequalities
   
   \[\frac{1}{4} < x < \frac{1}{2}(e^2 + 1)\] Multiplication Property of Inequalities
   
   \[0.25 < x < 2.0973\] Use a calculator.

Exercises

Solve each equation or inequality.

1. \(e^x = 120\)
   
   \[x \approx \ln 120\]

2. \(e^x = 25\)
   
   \[x = \ln 25\]

3. \(e^{-x} + 4 = 21\)
   
   \[x = -\ln 17\]

4. \(\ln 6x \geq 4\)
   
   \[x \geq \frac{e^4}{6}\]

5. \(\ln (x + 3) - 5 = -2\)
   
   \[x = -1\]

6. \(e^{3x} \leq 50\)
   
   \[x \leq \frac{\ln 50}{3}\]

7. \(e^{2x} - 1 - 3 = 12\)
   
   \[x = \frac{13}{2}\]

8. \(\ln (5x + 3) = 3.6\)
   
   \[x = \frac{e^{3.6} - 3}{5}\]

9. \(e^{3x} + 5 = 2\)
   
   \[x = -\frac{3}{3}\]

10. \(6 + 3e^{x^2} = 21\)
    
    \[x \approx \frac{1}{2}\ln 5\]

11. \(\ln (2x - 5) = 8\)
    
    \[x = \frac{e^8 + 5}{2}\]

12. \(\ln 5x + \ln 3x > 9\)
    
    \[x \approx 2.3243\]

Use a calculator to evaluate each expression to four decimal places.

1. \(e^1 = 2.0855\)
2. \(e^2 = 1.353\)
3. \(\ln 2 = 0.6931\)
4. \(\ln 0.99 = -2.079\)

Write an equivalent exponential or logarithmic equation.

5. \(e^2 - 3x = \ln 3\)
6. \(e^1 - 8x = 4\)

7. \(\ln 15 = x\)
8. \(\ln x - 0.6931 = 15\)

Evaluate each expression.

9. \(e^3 \cdot 3\)
10. \(e^{2x} \cdot 2x\)

11. \(\ln e^{-2.5} = -2.5\)
12. \(\ln e^y = y\)

Solve each equation or inequality.

13. \(e^{3x} = 5\)
14. \(e^x < 3.1\)

15. \(2e^x - 1 = 11\)
16. \(5e^x + 3 = 18\)

17. \(e^{3x} = 30\)
18. \(e^{4x} > 10\)

19. \(e^{3x} + 4 > 34\)
20. \(1 - 2e^{2x} = -19\)

21. \(\ln 3x = 2\)
22. \(\ln 8x = 3\)

23. \(\ln (x - 2) = 2\)
24. \(\ln (x + 3) - 1 = -0.2817\)

25. \(\ln (x + 3) = 4\)
26. \(\ln x + \ln 2x = 2\)
Chapter 9

9-5 Practice
Base e and Natural Logarithms

Use a calculator to evaluate each expression to four decimal places.

1. $e^{1.3} = 4.8417$
2. $\ln 8 = 2.0794$
3. $\ln 3.2 = 1.1632$
4. $e^{-0.6} = 0.5488$

5. $e^{1.2} = 66.8683$
6. $\ln 1 = 0$
7. $e^{-2.5} = 0.0821$
8. $\ln 0.007 = -3.2968$

Write an equivalent exponential or logarithmic equation.

9. $\ln 50 - x$
10. $5 \ln 36 - 2x$
11. $\ln 6 - 1.7918$
12. $\ln 9.3 - 2.3000$

3. $e^x = 50$
4. $e^{2x} = 36$
5. $e^{0.7918} - 6$
6. $e^{2.3000} = 9.3$

13. $e^x = 8$
14. $e^3 = 10x$
15. $e^{-x} = 4$
16. $e^x = x + 1$

$x = \ln 8$
5. $x = \ln 10x$
$x = -\ln 4$
2. $2 = \ln (x + 1)$

Evaluate each expression.

17. $e^{2.12} = 12$
18. $e^{3.5x} = 3x$
19. $\ln e^1 = -1$
20. $\ln e^{-2y} = -2y$

Solve each equation or inequality.

21. $e^x < 9$
22. $e^{-x} = 31$
23. $e^x = 1.1$
24. $e^x = 5.8$

$(x|x < 2.1972) = -3.4340$
$(x|x > 0.0953) = 1.7579$

25. $2e^x - 3 = 1$
26. $5e^x + 1 = 7$
27. $4 + e^t = 19$
28. $-3e^t + 10 < 8$

0.6931
$(x|x > 0.1823) = 2.7081$
$(x|x < -0.4055) = 0.4970$

29. $e^{3x} = 8$
30. $e^{-4x} = 5$
31. $e^{0.5x} = 6$
32. $2e^x = 24$

0.6931
$-0.4024$
$3.5835$
$0.4970$

33. $e^{2x} = 55$
34. $e^{3x} - 5 = 32$
35. $9 + e^{2x} = 10$
36. $e^{0.5x} + 7 = 15$

1.9945
$1.2036$
$0$
$0.8931$

37. $\ln 4x = 3$
38. $\ln (-2x) = 7$
39. $\ln 2.5x = 10$
40. $\ln (x - 6) = 1$

5.0214
$-548.3166$
$881.5863$
$8.7183$

41. $\ln (x + 2) = 3$
42. $\ln (x + 3) = 5$
43. $\ln 3x + \ln 2x = 9$
44. $\ln 5x + \ln x = 7$

18.0855
$145.4132$
$36.7493$
$14.8097$

INVESTING For Exercises 45 and 46, use the formula for continuously compounded interest, $A = P\text{e}^{rt}$, where $P$ is the principal, $r$ is the annual interest rate, and $t$ is the time in years.

45. If Sarita deposts $1000 in an account paying 3.4% annual interest compounded continuously, what is the balance in the account after 5 years? $\$1185.30$

46. How long will it take the balance in Sarita’s account to reach $\$2000? about 20.4 yr

47. RADIOACTIVE DECAY The amount of a radioactive substance $y$ that remains after $t$ years is given by the equation $y = ae^{kt}$, where $a$ is the initial amount present and $k$ is the decay constant for the radioactive substance. If $a = 100$, $y = 50$, and $k = -0.035$, find $t$. about 19.8 yr

Chapter 9

9-5 Word Problem Practice
Base e and Natural Logarithms

1. INTEREST Horatio opens a bank account that pays 2.3% annual interest compounded continuously. He makes an initial deposit of 10,000. What will be the balance of the account in 10 years? Assume that he makes no additional deposits and no withdrawals.

$12,586$

2. INTEREST Janie’s bank pays 2.8% annual interest compounded continuously on savings accounts. She placed $2000 in the account. How long will it take for her initial deposit to double in value? Assume that she makes no additional deposits and no withdrawals. Round your answer to the nearest quarter year.

24.75 yr

3. BACTERIA A bacterial population grows exponentially, doubling every 72 hours. Let $P$ be the initial population size and let $t$ be time in hours. Write a formula using the natural base exponential function that gives the size of the population as a function of $P$ and $t$.

$P = p(\text{e}^{\frac{t}{72}})$

4. POPULATION The equation $A = A_0\text{e}^{rt}$ describes the growth of the world’s population where $A$ is the population at time $t$, $A_0$ is the population at $t = 0$, and $r$ is the annual growth rate. How long will it take a population of 6.5 billion to increase to 9 billion if the annual growth rate is 2%?

16.3 yr

MONEY MANAGEMENT For Exercises 5-7, use the following information.

Linda wants to invest $20,000. She is looking at two possible accounts. Account A is a standard savings account that pays 3.4% annual interest compounded continuously. Account B would pay her a fixed amount of $200 every quarter.

5. If Linda can invest the money for 5 years only, which account would give her the higher return on her investment? How much more money would she make by choosing the higher paying account?

Account B: she’ll make $2400 – $23706.10 = $293.90 more

6. If Linda can invest the money for 10 years only, which account would give her the higher return on her investment? How much more money would she make by choosing the higher paying account?

Account A: she’ll make $35988.95 – $37000 = $98.55 more

7. If Linda can invest the money for 20 years only, which account would give her the higher return on her investment? How much more money would she make by choosing the higher paying account?

Account A: she’ll make $39477.55 – $36000 = $3477.55 more
Approximations for \( \pi \) and \( e \)

The following expression can be used to approximate \( e \). If greater and greater values of \( n \) are used, the value of the expression approximates \( e \) more and more closely.

\[
1 + \frac{1}{n} + \frac{\frac{1}{n}}{2!} + \frac{\frac{1}{n}}{3!} + \frac{\frac{1}{n}}{4!} + \ldots + \frac{\frac{1}{n}}{n!} \approx e 
\]

Another way to approximate \( e \) is to use this infinite sum. The greater the value of \( n \), the closer the approximation.

\[
e = 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \ldots + \frac{1}{2 \cdot 3 \cdot 4 \cdot \ldots \cdot n} + \ldots
\]

In a similar manner, \( \pi \) can be approximated using an infinite product discovered by the English mathematician John Wallis (1616–1703).

\[
\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \ldots \approx 2n - 1 \cdot 2n + 1
\]

Solve each problem.

1. Use a calculator with an \( e^x \) key to find \( e \) to 7 decimal places. \( 2.7182818 \)

2. Use the expression \((1 + \frac{1}{n})^n\) to approximate \( e \) to 3 decimal places. Use 5, 1000, 5000, and 7000 as values of \( n \). \( 2.488, 2.705, 2.716, 2.718 \)

3. Use the infinite sum to approximate \( e \) to 3 decimal places. Use the whole numbers from 3 through 6 as values of \( n \). \( 2.667, 2.708, 2.717, 2.718 \)

4. Which approximation method approaches the value of \( e \) more quickly? the infinite sum

5. Use a calculator with a \( \pi \) key to find \( \pi \) to 7 decimal places. \( 3.1415927 \)

6. Use the infinite product to approximate \( \pi \) to 3 decimal places. Use the whole numbers from 3 through 6 as values of \( n \). \( 2.926, 2.972, 3.002, 3.023 \)

7. Does the infinite product give good approximations for \( \pi \) quickly? no

8. Show that \( \pi^4 + \pi^5 \) is equal to \( e^6 \) to 4 decimal places. To 4 decimal places, they both equal 403.4288.

9. Which is larger, \( e^\pi \) or \( \pi^e \)? \( e^\pi > \pi^e \)

10. The expression \( x \) reaches a maximum value at \( x = e \). Use this fact to prove the inequality you found in Exercise 9.

\[
\frac{1}{e^x} > \frac{1}{x} > \frac{\ln x}{e} > \left( \frac{\ln x}{x} \right) > e^x > \pi^e
\]

Remember What You Learned

3. Visualizing their graphs is often a good way to remember the difference between mathematical equations. How can you know your knowledge of the graphs of exponential equations from Lesson 9-1 help you to remember that equations of the form \( y = a(1 + r)^t \) represent exponential growth, while equations of the form \( y = a(1 - r)^t \) represent exponential decay?

Sample answer: If \( a > 0 \), the graph of \( y = ab^x \) is always increasing if \( b > 1 \) and is always decreasing if \( 0 < b < 1 \). Since \( r \) is always a positive number, if \( b = 1 + r \), the base will be greater than 1 and the function will be increasing (growth), while if \( b = 1 - r \), the base will be less than 1 and the function will be decreasing (decay).
Study Guide and Intervention
Exponential Growth and Decay

Exponential Growth and Decay
Population increase and growth of bacteria colonies are examples of exponential growth. When a quantity increases by a fixed percent each time period, the amount of that quantity after $t$ time periods is given by $y = a(1 + r)^t$, where $a$ is the initial amount and $r$ is the percent increase expressed as a decimal.

Exponential Decay
Depreciation of value and radioactive decay are examples of exponential decay. When a quantity decreases by a fixed percent each time period, the amount of that quantity after $t$ time periods is given by $y = a(1 - r)^t$, where $a$ is the initial amount and $r$ is the percent decrease expressed as a decimal.

$y = a(1 - r)^t$  
$t = \frac{\log \left( \frac{y}{a} \right)}{\log (1 - r)}$

Example 1
A computer engineer is hired for a salary of $28,000. If she gets a 5% raise each year, after how many years will she be making over $50,000 or more?

Use the exponential growth model with $a = 28,000$, $y = 50,000$, and $r = 0.05$ and solve for $t$.

$50,000 = 28,000(1 + 0.05)^t$

$\log \left( \frac{50,000}{28,000} \right) = \log (1.05)^t$

$5 = t \log 1.05$

$t = 11.9$ years

If raises are given annually, she will be making over $50,000 in 12 years.

Exercises

1. BUSINESS A furniture store is closing out its business. Each week the owner lowers prices by 25%. After how many weeks will the sale price of a $500 item drop below $100?

6 weeks

2. INVESTMENT Carl plans to invest $500 at 8.25% interest, compounded continuously. How long will it take for his money to triple? about 14 years

3. SCHOOL POPULATION There are currently 850 students at the high school, which represents full capacity. The town plans an addition to house 400 more students. If the school population grows at 7.8% per year, in how many years will the new addition be full? about 5 years

4. EXERCISE Hugo begins a walking program by walking $\frac{3}{2}$ mile per day for one week. Each week thereafter he increases his mileage by 10%. After how many weeks is he walking more than 5 miles per day? 24 weeks

5. VOCABULARY GROWTH When Emily was 18 months old, she had a 10-word vocabulary. By the time she was 5 years old (60 months), her vocabulary was 2500 words. If her vocabulary increased at a constant percent per month, what was that increase? about 14%

Chapter 9

NAME __________________________ DATE ______________ PERIOD _____

Exponential Growth and Decay

A20

NAME __________________________ DATE ______________ PERIOD _____

Exponential Growth and Decay (continued)

Exponential Growth
Population increase and growth of bacteria colonies are examples of exponential growth. When a quantity increases by a fixed percent each time period, the amount of that quantity after $t$ time periods is given by $y = a(1 + r)^t$, where $a$ is the initial amount and $r$ is the percent increase (or rate of growth) expressed as a decimal.

Another exponential growth model often used by scientists is $y = aekt$, where $k$ is a constant.

Example 2
A certain strain of bacteria grows from 40 to 326 in 120 minutes. How long will it take for his money to triple? about 14 years

INVESTMENT

Carl plans to invest $500 at 8.25% interest, compounded continuously. How long will it take for his money to triple? about 14 years

VOCABULARY GROWTH

When Emily was 18 months old, she had a 10-word vocabulary. By the time she was 5 years old (60 months), her vocabulary was 2500 words. If her vocabulary increased at a constant percent per month, what was that increase? about 14%

Chapter 9

NAME __________________________ DATE ______________ PERIOD _____

Exponential Growth and Decay

A20

NAME __________________________ DATE ______________ PERIOD _____

Exponential Growth and Decay (continued)

Exponential Growth
Population increase and growth of bacteria colonies are examples of exponential growth. When a quantity increases by a fixed percent each time period, the amount of that quantity after $t$ time periods is given by $y = a(1 + r)^t$, where $a$ is the initial amount and $r$ is the percent increase (or rate of growth) expressed as a decimal.

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**Skills Practice**

**Exponential Growth and Decay**

1. **FISHING** In an over-fished area, the catch of a certain fish is decreasing at an average rate of 8% per year. If this decline persists, how long will it take for the catch to reach half of the amount before the decline? **about 8.3 yr**

2. **INVESTING** Alex invests $2000 in an account that has a 6% annual rate of growth. To the nearest year, when will the investment be worth $3600? **10 yr**

3. **POPULATION** A current census shows that the population of a city is 3.5 million. Using the formula \( P = a e^{rt} \), find the expected population of the city in 30 years if the growth rate \( r \) of the population is 1.5% per year, \( a \) represents the current population in millions, and \( t \) represents the time in years. **about 5.5 million**

4. **POPULATION** The population \( P \) in thousands of a city can be modeled by the equation \( P = 120000 e^{0.015t} \), where \( t \) is the time in years. In how many years will the population of the city be 120,000? **about 27 yr**

5. **BACTERIA** How many days will it take a culture of bacteria to increase from 2000 to 50,000 if the growth rate per day is 93.2%? **about 4.9 days**

6. **NUCLEAR POWER** The element plutonium-239 is highly radioactive. Nuclear reactors can produce and also use this element. The heat that plutonium-239 emits has helped to power equipment on the moon. If the half-life of plutonium-239 is 24,360 years, what is the value of 10 for this element? **about 0.00002845**

7. **DEPRECIATION** A Global Positioning Satellite (GPS) system uses satellite information to locate ground position. Abu’s surveying firm bought a GPS system for $12,500. The GPS depreciated by a fixed rate of 6% and is now worth $8600. How long ago did Abu buy the GPS system? **about 6.0 yr**

8. **BIOLOGY** In a laboratory, an organism grows from 100 to 250 in 8 hours. What is the hourly growth rate in the growth formula \( y = a(1 + rt) \)? **about 12.13%**

9. **INVESTING** The formula \( A = P \left(1 + \frac{r}{2}\right)^n\) gives the value of an investment after \( t \) years in an account that earns an annual interest rate \( r \) compounded twice a year. Suppose $500 is invested at 6% annual interest compounded twice a year. In how many years will the investment be worth $1000? **about 11.7 yr**

10. **BACTERIA** How many hours will it take a culture of bacteria to increase from 20 to 2000 if the growth rate per hour is 85%? **about 7.5 h**

11. **RADIOACTIVE DECAY** A radioactive substance has a half-life of 32 years. Find the constant \( k \) in the decay formula for the substance. **about 0.02166**

12. **DEPRECIATION** A piece of machinery valued at $250,000 depreciates at a fixed rate of 12% per year. After how many years will the value have depreciated to $100,000? **about 7.2 yr**

13. **INFLATION** For Dave to buy a new car comparably equipped to the one he bought 8 years ago would cost $12,500. Since Dave bought the car, the inflation rate for cars like his has been at an average annual rate of 5.1%. If Dave originally paid $8400 for the car, how long ago did he buy it? **about 8 yr**

14. **RADIOACTIVE DECAY** Cobalt, an element used to make alloys, has several isotopes. One of these, cobalt-60, is radioactive and has a half-life of 5.7 years. Cobalt-60 is used to trace the path of nonradioactive substances in a system. What is the value of 10 for Cobalt-60? **about 0.1216**

15. **WHALES** Modern whales appeared 5–10 million years ago. The vertebrae of a whale discovered by paleontologists contain roughly 0.25% as much carbon-14 as they would have contained when the whale was alive. How long ago did the whale die? Use \( k = 0.00012 \). **about 50,000 yr**

16. **POPULATION** The population of rabbits in an area is modeled by the growth equation \( P(t) = 80e^{0.0026t} \), where \( P \) is in thousands and \( t \) is in years. How long will it take for the population to reach 25,000? **about 4.4 yr**

17. **DEPRECIATION** A computer system depreciates at an average rate of 4% per month. If the value of the computer system was originally $12,000, in how many months is it worth $7350? **about 12 mo**

18. **BIOLOGY** In a laboratory, a culture increases from 30 to 195 organisms in 5 hours. What is the hourly growth rate in the growth formula \( y = a(1 + r)^t \)? **about 45.4%**
1. PROGRAMMING
For reasons having to do with speed, a computer programmer wishes to model population size using a natural base exponential function. However, the programmer is told that users of the program will be thinking in terms of the annual percentage increase. Let \( r \) be the percentage that the population increases each year. Find the value for \( k \) in terms of \( r \) so that

\[ e^k = 1 + r. \]

2. CARBON DATING
Archeologists uncover an ancient wooden tool. They analyze the tool and find that it has 22% as much carbon-14 compared to the likely amount that it contained when it was made. Given that the half-life of carbon-14 is about 5730 years, about how old is the artifact? Round your answer to the nearest 100 years.

3. POPULATION
The doubling time of a population is \( d \) years. The population size can be modeled by an exponential equation of the form \( Pe^{kt} \), where \( P \) is the initial population size and \( t \) is time. What is \( k \) in terms of \( d \)?

\[ k = \frac{\ln(2)}{d} \]

4. POPULATION
Louisa read that the population of her town has increased steadily at a rate of 2% each year. Today, the population of her town has grown to 68,735. Based on this information, what was the population of her town 100 years ago?

About 9,488 people.

CONSUMER AWARENESS
For Exercises 5–7, use the following information.

Jason wants to buy a brand new high-definition (HD) television. He could buy one now because he has $7000 to spend, but he thinks that if he waits, the quality of HD televisions will improve. His $7000 earns 2.5% interest annually compounded continuously. The television he wants to buy costs $5000 now, but the cost increases each year by 7%.

5. Write a natural base exponential function that gives the value of Jason's account as a function of time \( t \).

\[ P(t) = 7000e^{0.025t} \]

6. Write a natural base exponential function that gives the cost of the television Jason wants as a function of time \( t \).

\[ C(t) = 5000e^{0.07t} \]

7. In how many years will the cost of the television exceed the value of the money in Jason's account? In other words, how much time does Jason have to decide whether he wants to buy the television? Round your answer to the nearest tenth of a year.

7.9 years

Effective Annual Yield
When interest is compounded more than once per year, the effective annual yield is higher than the annual interest rate. The effective annual yield, \( E \), is the interest rate that would give the same amount of interest if the interest were compounded once per year. If \( P \) dollars are invested for one year, the value of the investment at the end of the year is \( A = P(1 + r)^n \). If \( P \) dollars are invested for one year at a nominal rate \( r \) compounded \( n \) times per year, the value of the investment at the end of the year is \( E = (1 + \frac{r}{n})^n - 1 \). Setting the amounts equal and solving for \( E \) will produce a formula for the effective annual yield.

\[ P(1 + E) = P(1 + \frac{r}{n})^n \]

If compounding is continuous, the value of the investment at the end of one year is \( A = Pe^{rt} \). Again set the amounts equal and solve for \( E \). A formula for the effective annual yield under continuous compounding is obtained.

\[ P(1 + E)e^{rt} = P(1 + \frac{r}{n})^n \]

Example 1
Find the effective annual yield of an investment made at 7.5% compounded monthly.

\[ r = 0.075 \]
\[ n = 12 \]
\[ E = (1 + \frac{0.075}{12})^{12} - 1 = 7.76% \]

Example 2
Find the effective annual yield of an investment made at 6.25% compounded continuously.

\[ r = 0.0625 \]
\[ E = e^{0.0625} - 1 \approx 6.45% \]

Exercises
Find the effective annual yield for each investment.

1. 10% compounded quarterly \( 10.38\% \)
2. 8.5% compounded monthly \( 8.84\% \)
3. 9.25% compounded continuously \( 9.69\% \)
4. 7.75% compounded continuously \( 8.06\% \)
5. 6.8% compounded daily (assume a 365-day year) \( 6.72\% \)
6. Which investment yields more interest—9% compounded continuously or 9.2% compounded quarterly? \( 9.2\% \) quarterly
Spreadsheet Activity

Net Present Value

You have learned how to use exponential and logarithmic functions to perform a number of financial analyses. Spreadsheets can be used to perform many types of analyses, such as calculating the Net Present Value of expenditures or investments. For example, when a business owner is considering a major purchase, it is a good idea to find out whether the investment will be profitable in the future. Consider the example of a local restaurant-delivery service that is debating whether to buy a used van for $8000. The owners of the company estimate that the van will bring in $2500 per year over four years. They can use the following formula to find the present value of the future cash flow to find the Net Present Value (NPV), that is, how much the profits would be worth in today’s dollars. NPV = \( \frac{CF_n}{(1 + r)^n} \), where \( CF_n \) is the cash flow in period \( n \) and \( r \) = the cost of capital, which is either the interest that will be paid on a loan or the interest that the money would earn if it were invested.

**Exercises**

1. If the NPV is greater than the cost, the investment will pay for itself. Based on the spreadsheet shown above, would it be cost-effective for the company to buy the van? Explain. The cost is actually about $75 more than the NPV, so it would not be cost-effective to buy the van.

2. Four times a year, Josey and Drew publish a magazine. They want to buy a color printer that costs $1750. The cost of capital for this purchase would be 6%. They are planning to raise the price of their magazine from $1 to $2. Create a spreadsheet to determine the NPV for this purchase.
   a. The last issue of the magazine sold 500 copies. If each issue of the magazine printed in color sells 100 copies more than the previous issue, is the printer a good investment after one year? Explain. No, after one year the NPV is only about $1682.14.
   b. If the sales of the magazine continue to rise at the same rate, is the printer a good investment after two years? Yes, after two years the NPV is about $5210.28. The NPV is about $3460.28 greater than the cost.

3. a. Calculate the NPV for an investment over a period of six years if the cost of capital is 4.5% and the investment will bring a cash flow of $750 every year. The NPV would be about $3868.40.
   b. Would this be a good investment of $3000? Explain? Yes, the NPV is $1131.60 greater than the cost.