Anticipation Guide

Matrices

STEP 1

Before you begin Chapter 4

- Read each statement.
- Decide whether you Agree (A) or Disagree (D) with the statement.
- Write A or D in the first column OR if you are not sure whether you agree or disagree, write NS (Not Sure).

<table>
<thead>
<tr>
<th>Statement</th>
<th>STEP 2 A or D</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. A matrix contains constants or variables in horizontal rows and vertical columns.</td>
<td>A</td>
</tr>
<tr>
<td>2. Each value in a matrix is called a term.</td>
<td>D</td>
</tr>
<tr>
<td>3. If two matrices contain the same numbers but have a different number of rows or columns, then they are not equal.</td>
<td>A</td>
</tr>
<tr>
<td>4. Two matrices with different dimensions can be added or subtracted by adding zeros so that both matrices have the same dimensions.</td>
<td>D</td>
</tr>
<tr>
<td>5. The product of a matrix and a constant can be found by multiplying each element of the matrix by that constant.</td>
<td>A</td>
</tr>
<tr>
<td>6. The associative, commutative, and distributive properties of multiplication are all true for matrices.</td>
<td>D</td>
</tr>
<tr>
<td>7. A translation is a transformation in which a figure is turned around a single point.</td>
<td>A</td>
</tr>
<tr>
<td>8. A vertex matrix is a matrix containing the coordinates of the vertices of a figure.</td>
<td>A</td>
</tr>
<tr>
<td>9. A third-order determinant of a matrix contains three columns and any number of rows.</td>
<td>D</td>
</tr>
<tr>
<td>10. Each element of an identity matrix for multiplication is 1.</td>
<td>D</td>
</tr>
<tr>
<td>11. Two matrices are inverses of each other if their product is the identity matrix.</td>
<td>A</td>
</tr>
<tr>
<td>12. To solve a system of equations using matrices, you must write a matrix for the coefficients, one for the variables, and one for the constants.</td>
<td>A</td>
</tr>
</tbody>
</table>

STEP 2

After you complete Chapter 4

- Reread each statement and complete the last column by entering an A or a D.
- Did any of your opinions about the statements change from the first column?
- For those statements that you mark with a D, use a piece of paper to write an example of why you disagree.

Lesson Reading Guide

Introduction to Matrices

Get Ready for the Lesson

Read the introduction to Lesson 4-1 in your textbook.

a. What is the base price of a Mid-Size SUV? $27,350
b. What is the exterior length of a Compact SUV? 175.2 in.

Read the Lesson

1. Give the dimensions of each matrix.
   a. \[
   \begin{pmatrix}
   3 & 2 & 5 \\
   -1 & 0 & 6
   \end{pmatrix}
   \] \(2 \times 3\)
   b. \[
   \begin{pmatrix}
   1 & 4 & 0 & -8 \\
   2 & 1 & 5
   \end{pmatrix}
   \] \(1 \times 5\)

2. Identify each matrix with as many of the following descriptions that apply: row matrix, column matrix, square matrix, zero matrix.
   a. \[
   \begin{pmatrix}
   6 & 5 & 4 & 3
   \end{pmatrix}
   \] row matrix
   b. \[
   \begin{pmatrix}
   0 \\
   0 \\
   0
   \end{pmatrix}
   \] column matrix; zero matrix
   c. \[
   \begin{pmatrix}
   0
   \end{pmatrix}
   \] row matrix; column matrix; square matrix; zero matrix

3. Write a system of equations that you could use to solve the following matrix equation for \(x\), \(y\), and \(z\). (Do not actually solve the system.)
   \[
   \begin{pmatrix}
   3x \\
   x + y \\
   y - z
   \end{pmatrix}
   \begin{pmatrix}
   -9 \\
   5 \\
   6
   \end{pmatrix}
   \]
   \(3x = -9, x + y = 5, y - z = 6\)

Remember What You Learned

4. Some students have trouble remembering which number comes first in writing the dimensions of a matrix. Think of an easy way to remember this.
   Sample answer: Read the matrix from top to bottom, then from left to right. Reading down gives the number of rows, which is written first in the dimensions of the matrix. Reading across gives the number of columns, which is written second.
### Study Guide and Intervention

#### Introduction to Matrices

**Organize Data**

A matrix can be described by its **dimensions**. A matrix with $m$ rows and $n$ columns is an $m \times n$ matrix.

#### Example 1

Owls' eggs incubate for 30 days and their fledgling period is also 30 days. Swifts' eggs incubate for 20 days and their fledgling period is 44 days. Eggs of the king penguin incubate for 53 days, and the fledgling time for a king penguin is 360 days. Write a $2 \times 4$ matrix to organize this information. 

**Owl**

<table>
<thead>
<tr>
<th>Animal</th>
<th>Incubation</th>
<th>Fledgling</th>
</tr>
</thead>
<tbody>
<tr>
<td>Swift</td>
<td>30</td>
<td>44</td>
</tr>
<tr>
<td>Pigeon</td>
<td>15</td>
<td>360</td>
</tr>
<tr>
<td>King Penguin</td>
<td>53</td>
<td>360</td>
</tr>
</tbody>
</table>

#### Example 2

What are the dimensions of matrix $A$ if $A = \left[ \begin{array}{ccc} 10 & 15 & 30 \\ 2 & 8 & 18 \\ 8 & 4 & 4 \end{array} \right]$?

Since matrix $A$ has 3 rows and 3 columns, the dimensions of $A$ are $3 \times 3$.

#### Exercises

**State the dimensions of each matrix.**

1. $\left[ \begin{array}{ccc} 15 & 22 & 8 \\ 14 & 7 & 24 \\ 68 & 3 & 28 \end{array} \right] \times \left[ \begin{array}{ccc} 4 \times 4 \\ 16 & 12 & 0 \end{array} \right] \times 3$

2. $\left[ \begin{array}{ccc} 71 & 44 \\ 38 & 27 \\ 92 & 53 \\ 26 & 65 \end{array} \right] \times \left[ \begin{array}{ccc} 5 \times 2 \end{array} \right]$

**A travel agent provides for potential travelers the normal high temperatures for the months of January, April, July, and October for various cities. In Boston these figures are $36^\circ$, $54^\circ$, $58^\circ$, and $68^\circ$. In Dallas they are $76^\circ$, $82^\circ$, and $79^\circ$. In Los Angeles they are $64^\circ$, $74^\circ$, and $60^\circ$, and in St. Louis they are $38^\circ$, $67^\circ$, $89^\circ$, and $63^\circ$. Organize this information in a $4 \times 5$ matrix.**

**Chapter 4**

**Answers (Lesson 4-1)**

- **Exercise 1:**
  - 1. $[5x + 4y] = [20 + 20]$ (20, 20)
  - 2. $[3x + 2y] = [18 + 20]$ (20, 20)
  - 3. $[2x + 3y] = [12 + 15]$ (15, 15)
  - 4. $[x + y] = [8 + 10]$ (10, 10)
  - 5. $[x + y] = [8 + 6]$ (6, 6)

- **Exercise 2:**
  - 1. $[x + y] = [2 + 2]$ (2, 2)
  - 2. $[x + y] = [3 + 4]$ (4, 4)
  - 3. $[x + y] = [1 + 2]$ (2, 2)

- **Exercise 3:**
  - 1. $[x + y] = [2 + 3]$ (3, 3)
  - 2. $[x + y] = [1 + 2]$ (2, 2)

- **Exercise 4:**
  - 1. $[x + y] = [3 + 4]$ (4, 4)
  - 2. $[x + y] = [2 + 3]$ (3, 3)
  - 3. $[x + y] = [1 + 2]$ (2, 2)
4 - 1 \hspace{1cm} \textbf{Skills Practice}

\textit{Introduction to Matrices}

State the dimensions of each matrix.

1. \[\begin{bmatrix} 3 & 2 & 4 \\ -1 & 4 & 0 \end{bmatrix}\] \(2 \times 3\)

2. \(\begin{bmatrix} 0 & 15 \end{bmatrix}\) \(1 \times 2\)

3. \(\frac{3}{2} \times 2\)

4. \(\frac{6}{7} \times 3\)

5. \(\frac{13}{4} \times 3\)

6. \(\begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 1 \end{bmatrix}\) \(2 \times 4\)

Solve each equation.

7. \((5x + 3y) = [15 \ 12]\) \((3, 4)\)

8. \([3x - 2y] = [7 \ 3]\)

9. \(\frac{7x}{14} - \frac{14}{2y} = (-2, 7)\)

10. \((2x - 8y + z) = [10 \ 16 \ -1]\) \((5, -2, -1)\)

11. \(\frac{8x}{2y} + \frac{1}{2} = (4, 3)\)

12. \(\frac{5x + 6y}{6 - 7} = \frac{10x}{32}\) \((2, 4)\)

13. \(\frac{3x + 2}{7y - 2} = \frac{3x - 2}{10}\) \((0, -2)\)

14. \(\frac{4x - 5}{9y + 5} = \frac{3x - 3}{1, -1}\)

15. \(\frac{3x + 1}{12} = \frac{7}{12} \times 4 - \frac{2}{20}\) \((2, 11, 7)\)

16. \(\frac{5x + 6y}{3x + 2} = \frac{1}{1} \times 3, 1\)

17. \(\frac{4x}{8z} = \frac{4 + 1}{4x} \times 1, 4, 0\)

18. \(\frac{4x + 3}{4x} = \frac{12}{3}\)

19. \(\frac{2x + 3}{6y} = (3, 1)\)

20. \(\frac{x}{3y} = \frac{3}{4} \times (12, 3)\)

14. \textbf{TICKET PRICES}

The table at the right gives ticket prices for a concert. Write a \(2 \times 3\) matrix that represents the cost of a ticket.

<table>
<thead>
<tr>
<th>Child</th>
<th>Student</th>
<th>Adult</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6</td>
<td>$12</td>
<td>$18</td>
</tr>
<tr>
<td>$8</td>
<td>$15</td>
<td>$22</td>
</tr>
</tbody>
</table>

\textbf{CONSTRUCTION}

For Exercises 15 and 16, use the following information.

During each of the last three weeks, a road-building crew has used three truckloads of gravel. The table at the right shows the amount of gravel in each load.

15. Write a matrix for the amount of gravel in each load.

\[\begin{bmatrix} 40 & 32 & 24 \\ 40 & 40 & 24 \\ 32 & 24 & 24 \end{bmatrix}\]

16. What are the dimensions of the matrix? \(3 \times 3\)
4-1 Word Problem Practice
Introduction to Matrices

1. HAWAII The table shows the population and area of some of the islands in Hawaii. What would be the dimensions of a matrix that represented this information?

<table>
<thead>
<tr>
<th>Island</th>
<th>Population</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hawaii</td>
<td>1,203,817</td>
<td>10,445</td>
</tr>
<tr>
<td>Maui</td>
<td>101,361</td>
<td>729</td>
</tr>
<tr>
<td>Oahu</td>
<td>896,231</td>
<td>594</td>
</tr>
<tr>
<td>Kauai</td>
<td>55,947</td>
<td>548</td>
</tr>
<tr>
<td>Lanai</td>
<td>2,426</td>
<td>140</td>
</tr>
</tbody>
</table>

Source: www.state.hawaii.gov

5 by 2 or 2 by 5

2. LAUNDRY Carl is looking for a Laundromat. SuperWash has 20 small washers, 10 large washers, and 20 dryers. QuickClean has 40 small washers, 5 large washers, and 50 dryers. ToughWash has 15 small washers, 40 large washers, and 100 dryers. Write a matrix to organize this information.

Sample answer:

\[
\begin{bmatrix}
20 & 10 & 20 \\
40 & 5 & 50 \\
15 & 40 & 100
\end{bmatrix}
\]

3. CITY DISTANCES The incomplete matrix shown gives the approximate distances between Chicago, Los Angeles, and New York City. Complete the matrix.

<table>
<thead>
<tr>
<th></th>
<th>NYC</th>
<th>Chicago</th>
<th>Los Angeles</th>
</tr>
</thead>
<tbody>
<tr>
<td>NYC</td>
<td>0</td>
<td>810</td>
<td>2,790</td>
</tr>
<tr>
<td>Chicago</td>
<td>810</td>
<td>0</td>
<td>2,050</td>
</tr>
<tr>
<td>Los Angeles</td>
<td>2,790</td>
<td>2,050</td>
<td>0</td>
</tr>
</tbody>
</table>

Sample answer:

\[
\begin{bmatrix}
0 & 810 & 2,790 \\
810 & 0 & 2,050 \\
2,790 & 2,050 & 0
\end{bmatrix}
\]

4. INVENTORY A store manager records the number of light bulbs in stock for 3 different brands over a five-day period. The manager decides to make a matrix of this information. Each row represents a different brand, and each column represents a different day. The entry in column \( N \) represents the inventories at the beginning of day \( N \).

\[
\begin{bmatrix}
25 & 24 & 22 & 20 & 19 \\
30 & 27 & 25 & 22 & 21 \\
28 & 25 & 21 & 19 & 19
\end{bmatrix}
\]

Assuming that the inventories were never replenished, which brand holds the record for most light bulbs sold on a given day? the third row brand

5. SHOE SALES For Exercises 5 and 6, use the following information.

A shoe store manager keeps track of the amount of money made by each of three salespeople for each day of a workweek. Monday through Friday, Carla made $40, $70, $35, $50, and $20. John made $30, $60, $20, $45, and $30. Mary made $35, $90, $30, $40, and $30.

Sample answer:

\[
\begin{bmatrix}
40 & 70 & 35 & 50 & 20 \\
30 & 60 & 20 & 45 & 30 \\
35 & 90 & 30 & 40 & 30
\end{bmatrix}
\]

6. Which salesperson made the most money that week? Mary

4-1 Enrichment
Matrices and Networks

Graph theory is a branch of mathematics that explores situations represented by points called vertices and the segments that may connect them, called edges. For example, graphs can be used to represent computer networks or airline routes between major cities.

An incidence matrix is a matrix used to represent the vertices, edges, and relationships among the vertices of a graph. The names each row and column. For example, the incidence matrix for the computer network shown in the figure is shown below.

The numbers represent how many edges connect the vertices.

Indicates one edge from \( C_1 \) to \( C_2 \)

Indicates no edge from \( C_4 \) to \( C_1 \)

Complete the incidence matrix for each pictured network.

1. Puddlejump Airlines daily flights between cities 1–5.

2. Computers in a Hub and Spoke network.
Add and Subtract Matrices

Addition of Matrices

\[
\begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix}
+ 
\begin{bmatrix}
    j & k \\
    l & m
\end{bmatrix}
= 
\begin{bmatrix}
    a+j & b+k \\
    c+l & d+m
\end{bmatrix}
\]

Subtraction of Matrices

\[
\begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix}
- 
\begin{bmatrix}
    j & k \\
    l & m
\end{bmatrix}
= 
\begin{bmatrix}
    a-j & b-k \\
    c-l & d-m
\end{bmatrix}
\]

Example 1

Find \( A + B \) if \( A = \begin{bmatrix} 6 & -7 \\ 3 & -12 \end{bmatrix} \) and \( B = \begin{bmatrix} 4 & 3 \\ -5 & -6 \end{bmatrix} \).

\[
A + B = \begin{bmatrix} 6 & -7 \\ 3 & -12 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -5 & -6 \end{bmatrix} = \begin{bmatrix} 10 & -4 \\ -2 & -18 \end{bmatrix}
\]

Example 2

Find \( A - B \) if \( A = \begin{bmatrix} -3 & -\frac{9}{10} \\ 10 & -\frac{7}{2} \end{bmatrix} \) and \( B = \begin{bmatrix} -3 & -\frac{3}{5} \\ 16 & -1 \end{bmatrix} \).

\[
A - B = \begin{bmatrix} -3 & -\frac{9}{10} \\ 10 & -\frac{7}{2} \end{bmatrix} - \begin{bmatrix} -3 & -\frac{3}{5} \\ 16 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{27}{10} \\ -6 & -\frac{3}{2} \end{bmatrix}
\]

Exercise

Perform the indicated operations. If the matrix does not exist, write impossible.

1. \[
\begin{bmatrix}
    8 & 7 \\
    -10 & -6
\end{bmatrix} + 
\begin{bmatrix} -4 & 3 \\ 2 & -12 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ -8 & -6 \end{bmatrix}
\]

2. \[
\begin{bmatrix} -6 & -5 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 9 \\ -4 & 5 \end{bmatrix} = \begin{bmatrix} -12 & -14 \\ -1 & 9 \end{bmatrix}
\]

3. \[
\begin{bmatrix} 5 & 0 \\ -6 & 3 \end{bmatrix} + \begin{bmatrix} -2 & -3 \\ 2 & 17 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 20 \end{bmatrix}
\]

4. \[
\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 6 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 5 & 4 \end{bmatrix}
\]

5. \[
\begin{bmatrix} 8 & 5 \\ -1 & -11 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 1 & 0 \end{bmatrix}
\]

6. \[
\begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 4 & 7 \end{bmatrix}
\]

Sample answer: A scale of miles tells you how to multiply the distances you measure on a map by a certain number to get the actual distance between two locations. This multiplier is often called a scale factor. A scalar represents the same idea: It is a real number by which a matrix can be multiplied to change all the elements of the matrix by a uniform scale factor.
**Answers (Lesson 4-2)**

### Scalar Multiplication

1. \( \frac{1}{2} \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} \)

2. \( 2 \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \)

3. \( \frac{1}{3} \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \)

4. \( -1 \begin{bmatrix} 5 & -3 \\ 6 & 0 \end{bmatrix} \)

5. \( 2 \begin{bmatrix} 5 & 0 & 2 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

6. \( 3 \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \)

7. \( \frac{1}{2} \begin{bmatrix} 4 & 0 \\ -2 & 6 \end{bmatrix} \)

8. \( -2 \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & -1 \end{bmatrix} \)

9. \( \frac{1}{3} \begin{bmatrix} 9 & 6 \\ 3 & 2 \end{bmatrix} \)

10. \( -\frac{1}{2} \begin{bmatrix} 8 & -4 \\ 0 & 6 \end{bmatrix} \)

11. \( -3 \begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix} \)

12. \( -\frac{1}{2} \begin{bmatrix} 8 & 1 \\ 6 & 0 \end{bmatrix} \)

13. \( 2 \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} \)

14. \( -2 \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \)

15. \( 3 \begin{bmatrix} 1 & -4 \\ 2 & 5 \end{bmatrix} \)

16. \( -4 \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} \)

17. \( -\frac{1}{3} \begin{bmatrix} 6 & 9 \\ -2 & 3 \end{bmatrix} \)

**Exercises**

Perform the indicated scalar multiplication. If the matrix does not exist, write impossible.

1. \( \frac{1}{2} \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} \)

2. \( 2 \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \)

3. \( \frac{1}{3} \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \)

4. \( -1 \begin{bmatrix} 5 & -3 \\ 6 & 0 \end{bmatrix} \)

5. \( 2 \begin{bmatrix} 5 & 0 & 2 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

6. \( 3 \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \)

7. \( \frac{1}{2} \begin{bmatrix} 4 & 0 \\ -2 & 6 \end{bmatrix} \)

8. \( -2 \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & -1 \end{bmatrix} \)

9. \( \frac{1}{3} \begin{bmatrix} 9 & 6 \\ 3 & 2 \end{bmatrix} \)

10. \( -\frac{1}{2} \begin{bmatrix} 8 & -4 \\ 0 & 6 \end{bmatrix} \)

11. \( -3 \begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix} \)

12. \( -\frac{1}{2} \begin{bmatrix} 8 & 1 \\ 6 & 0 \end{bmatrix} \)

13. \( 2 \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} \)

14. \( -2 \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \)

15. \( 3 \begin{bmatrix} 1 & -4 \\ 2 & 5 \end{bmatrix} \)

16. \( -4 \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} \)

17. \( -\frac{1}{3} \begin{bmatrix} 6 & 9 \\ -2 & 3 \end{bmatrix} \)
4-2
Practice

Operations with Matrices

Perform the indicated matrix operations. If the matrix does not exist, write impossible.

1. \[
\begin{bmatrix}
2 & -1 \\
3 & 4 \\
\end{bmatrix}
+ \begin{bmatrix}
-6 & 9 \\
-12 & 11 \\
\end{bmatrix}
= \begin{bmatrix}
4 & 8 \\
-9 & 25 \\
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
4 & 3 \\
2 & 1 \\
\end{bmatrix}
- \begin{bmatrix}
-6 & 8 \\
-2 & 6 \\
\end{bmatrix}
= \begin{bmatrix}
10 & 11 \\
4 & 7 \\
\end{bmatrix}
\]

3. \[
\begin{bmatrix}
-3 \\
17 \\
\end{bmatrix}
+ \begin{bmatrix}
0 \\
4 \\
\end{bmatrix}
= \begin{bmatrix}
-3 \\
21 \\
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
9 \\
135 \\
\end{bmatrix}
+ 5 \begin{bmatrix}
2 \\
7 \\
\end{bmatrix}
= \begin{bmatrix}
16 \\
154 \\
\end{bmatrix}
\]

5. \[
\begin{bmatrix}
2 \\
-11 \\
\end{bmatrix}
+ \begin{bmatrix}
9 \\
16 \\
\end{bmatrix}
= \begin{bmatrix}
11 \\
5 \\
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
8 \\
4 \\
\end{bmatrix}
- \begin{bmatrix}
12 \\
-16 \\
\end{bmatrix}
= \begin{bmatrix}
-4 \\
20 \\
\end{bmatrix}
\]

Use \(A = \begin{bmatrix}
4 & -3 & 0 \\
-1 & 6 & 2 \\
\end{bmatrix}\), \(B = \begin{bmatrix}
-2 & 4 & 5 \\
1 & 0 & 9 \\
\end{bmatrix}\), and \(C = \begin{bmatrix}
10 & -6 \\
-4 & 5 \\
\end{bmatrix}\) to find the following.

7. \(A - B = \begin{bmatrix}
-1 & -5 & -5 \\
-3 & 6 & 11 \\
\end{bmatrix}\)

8. \(A - C = \begin{bmatrix}
-4 & 7 & -6 \\
10 & -18 \\
\end{bmatrix}\)

9. \(-3B = \begin{bmatrix}
-6 & -12 & -15 \\
9 & 0 & 27 \\
\end{bmatrix}\)

10. \(4A - B = \begin{bmatrix}
-12 & -17 & 20 \\
-9 & 6 & -38 \\
\end{bmatrix}\)

11. \(-2B - 3C = \begin{bmatrix}
-28 & -16 & -28 \\
-36 & 12 & -42 \\
\end{bmatrix}\)

12. \(A + 0.5C = \begin{bmatrix}
-5 & 14 & 12 \\
9 & 6 & 12 \\
\end{bmatrix}\)

ECONOMICS For Exercises 13 and 14, use the table that shows loans by an economic development board to women and men starting new businesses.

<table>
<thead>
<tr>
<th></th>
<th>Women</th>
<th></th>
<th>Men</th>
</tr>
</thead>
<tbody>
<tr>
<td>Businesses</td>
<td></td>
<td></td>
<td></td>
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<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Women</th>
<th></th>
<th>Men</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount ($)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

13. Write two matrices that represent the number of new businesses and loan amounts, one for women and one for men.

14. Find the sum of the numbers of new businesses and loan amounts for both men and women over the three-year period expressed as a matrix.

15. PET NUTRITION Use the table that gives nutritional information for two types of dog food. Find the difference in the percent of protein, fat, and fiber between Mix B and Mix A expressed as a matrix.

<table>
<thead>
<tr>
<th></th>
<th>% Protein</th>
<th>% Fat</th>
<th>% Fiber</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mix A</td>
<td>22</td>
<td>12</td>
<td>5</td>
</tr>
<tr>
<td>Mix B</td>
<td>24</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

Chapter 4
**Sundaram’s Sieve**

The properties and patterns of prime numbers have fascinated many mathematicians. In 1934, a young East Indian student named Sundaram constructed the following matrix.

<table>
<thead>
<tr>
<th>4</th>
<th>7</th>
<th>10</th>
<th>13</th>
<th>16</th>
<th>19</th>
<th>22</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>12</td>
<td>17</td>
<td>22</td>
<td>27</td>
<td>32</td>
<td>37</td>
<td>42</td>
</tr>
<tr>
<td>10</td>
<td>17</td>
<td>24</td>
<td>31</td>
<td>38</td>
<td>45</td>
<td>52</td>
<td>59</td>
</tr>
<tr>
<td>13</td>
<td>22</td>
<td>31</td>
<td>40</td>
<td>49</td>
<td>58</td>
<td>67</td>
<td>76</td>
</tr>
<tr>
<td>16</td>
<td>27</td>
<td>38</td>
<td>49</td>
<td>60</td>
<td>71</td>
<td>82</td>
<td>93</td>
</tr>
<tr>
<td>19</td>
<td>30</td>
<td>42</td>
<td>54</td>
<td>66</td>
<td>78</td>
<td>90</td>
<td>102</td>
</tr>
<tr>
<td>22</td>
<td>34</td>
<td>46</td>
<td>58</td>
<td>70</td>
<td>82</td>
<td>94</td>
<td>106</td>
</tr>
<tr>
<td>25</td>
<td>38</td>
<td>51</td>
<td>64</td>
<td>77</td>
<td>90</td>
<td>103</td>
<td>116</td>
</tr>
</tbody>
</table>

A surprising property of this matrix is that it can be used to determine whether or not some numbers are prime.

**Complete these problems to discover this property.**

1. The first row and the first column are created by using an arithmetic pattern. What is the common difference used in the pattern? 3

2. Find the next four numbers in the first row. 28, 31, 34, 37

3. What are the common differences used to create the patterns in rows 2, 3, 4, and 5? 5, 7, 9, 11

4. Write the next two rows of the matrix. Include eight numbers in each row.
   - Row 6: 19, 32, 45, 58, 71, 84, 97, 110
   - Row 7: 22, 37, 52, 67, 82, 97, 112, 127

5. Choose any five numbers from the matrix. For each number \( n \), that you chose from the matrix, find \( 2n + 1 \). Answers will vary.

6. Write the factorization of each value of \( 2n + 1 \) that you found in problem 5. Answers will vary, but all numbers are composite.

7. Use your results from problems 5 and 6 to complete this statement: If \( n \) occurs in the matrix, then \( 2n + 1 \) is not a prime number. (is/is not) a prime number.

8. Choose any five numbers that are not in the matrix. Find \( 2n + 1 \) for each of these numbers. Show that each result is a prime number. Answers will vary, but all numbers are prime.

9. Complete this statement: If \( n \) does not occur in the matrix, then \( 2n + 1 \) is a prime number.

**Answers will vary.**
For any matrices

\[ AB = [\begin{array}{cc} a_1 & a_2 \\ b_1 & b_2 \end{array}] \]

\[ BC = [\begin{array}{cc} c_1 & c_2 \\ d_1 & d_2 \end{array}] \]

The Commutative Property of Multiplication does not hold for matrices.

\[ AB \neq BA \]

Example

Find each product, if possible.

1. \[ 4 \quad 1 \]
   \[ 3 \quad 0 \]
   \[ 2 \quad 3 \]

2. \[ -1 \quad 0 \]
   \[ -3 \quad 2 \]
   \[ -3 \quad 2 \]

3. \[ 3 \quad -1 \]
   \[ 2 \quad 4 \]
   \[ 3 \quad -1 \]

4. \[ -3 \quad 1 \]
   \[ 4 \quad 0 \]
   \[ -2 \quad -3 \]

5. \[ 3 \quad 2 \]
   \[ 0 \quad 4 \]
   \[ 1 \quad 2 \]

6. \[ 5 \quad -2 \]
   \[ 2 \quad -3 \]
   \[ 4 \quad -1 \]

7. \[ 6 \quad 10 \]
   \[ -2 \quad -2 \]
   \[ 0 \quad 4 \]

8. \[ 7 \quad 2 \]
   \[ -2 \quad 0 \]
   \[ -3 \quad 0 \]

9. \[ 2 \quad 0 \]
   \[ -3 \quad -3 \]
   \[ -1 \quad -2 \]

not possible

Exercises

Find each product, if possible.

1. \[ 4 \quad 1 \]
   \[ 3 \quad 0 \]

2. \[ -1 \quad 0 \]
   \[ -3 \quad 2 \]

3. \[ 3 \quad -1 \]
   \[ 2 \quad 4 \]

4. \[ -3 \quad 1 \]
   \[ 4 \quad 0 \]

5. \[ 3 \quad 2 \]
   \[ 0 \quad 4 \]

6. \[ 5 \quad -2 \]
   \[ 2 \quad -3 \]

7. \[ 6 \quad 10 \]
   \[ -2 \quad -2 \]

8. \[ 7 \quad 2 \]
   \[ -2 \quad 0 \]

9. \[ 2 \quad 0 \]
   \[ -3 \quad -3 \]

not possible

Use \[ A = [\begin{array}{cc} a_1 & a_2 \\ b_1 & b_2 \end{array}] \]

\[ B = [\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}] \]

\[ C = [\begin{array}{cc} 1 & 2 \\ 1 & 3 \end{array}] \]

and scalar \( c = -4 \) to determine whether each of the following equations is true for the given matrices.

1. \( cAB = (cA)B \) yes

2. \( AB = BA \) no

3. \( BC = CB \) no

4. \( (AB)C = A(BC) \) yes

5. \( C(A + B) = AC + BC \) no

6. \( c(A + B) = cA + cB \) yes
### 4-3 Skills Practice

**Multiplying Matrices**

Determine whether each matrix product is defined. If so, state the dimensions of the product.

1. \(A_2 \times 5 \cdot B_3 \times 1 \) = undefined

2. \(M_1 \times 3 \cdot N_3 \times 2 \) = undefined

3. \(B_3 \times 2 \cdot A_3 \times 2 \) = undefined

4. \(R_4 \times 4 \cdot S_4 \times 1 \) = undefined

5. \(X_3 \times 3 \cdot Y_3 \times 4 \) = undefined

6. \(A_6 \times 4 \cdot B_4 \times 5 \) = undefined

Find each product, if possible.

7. \([2 \ 1] \cdot [2] = 8\)

8. \([\frac{5}{2} \ \ 6] \cdot [\frac{2}{3} \ -1] = [\frac{28}{7} \ -9]\)

9. \([\frac{1}{3} \ 2] \cdot [\frac{3}{-1} \ 1] = \frac{-3}{-5}\)

10. \([\frac{3}{-2} \ 1] \cdot [\frac{1}{3} \ 1] = \frac{28}{7} \ -9\)

11. \([1 \ 2] \cdot [9 \ -1] = [8 \ 11]\)

12. \([\frac{1}{3}] \cdot [2 \ -3 \ 3 \ 2] = [\frac{-2}{6} \ 3 \ 2]\)

13. \([\frac{5}{6} \ -\frac{4}{3}] \cdot [\frac{3}{1} \ 0] = [\frac{-12}{-6} \ 20 \ 6 \ 0]\)

14. \([\frac{2}{3} \ -\frac{2}{5}] \cdot [\frac{0}{3} \ 1] = [\frac{-6}{15} \ 6 \ 12]\)

15. \([\frac{-1}{2} \ \frac{3}{3}] \cdot [\frac{-3}{0} \ 2] = [\frac{-12}{-6} \ 20 \ 6 \ 0]\)

16. \([\frac{1}{1} \ \frac{1}{1}] \cdot [\frac{2}{2} \ 2] = [\frac{2}{4} \ 4]\)

Use \(A = [\frac{2}{3} \ \frac{1}{1}]\), \(B = [\frac{2}{-3} \ \frac{1}{2}]\), and scalar \(c = 2\) to determine whether the following equations are true for the given matrices.

17. \((AC)c = A(Cc)\) yes

18. \(AB = BA\) no

19. \(B(A + C) = AB + BC\) no

20. \((A - B)c = Ac - Bc\) yes

### 4-3 Practice

**Multiplying Matrices**

Determine whether each matrix product is defined. If so, state the dimensions of the product.

1. \(A_4 \times 7 \cdot B_7 \times 3\)

2. \(A_3 \times 5 \cdot M_5 \times 3\)

3. \(M_2 \times 1 \cdot A_4 \times 6\)

4. \(M_3 \times 2 \cdot A_3 \times 2\) undefined

5. \(R_1 \times 9 \cdot Q_9 \times 1\)

6. \(P_9 \times 1 \cdot Q_9 \times 9\)

Find each product, if possible.

7. \([2 \ 3] \cdot [\frac{3}{3} \ -2 \ 7] = [30 \ -8 \ 26]\)

8. \([\frac{2}{3} \ -1] \cdot [\frac{2}{3} \ 0 \ -1] = [\frac{-2}{23} \ 20]\)

9. \([\frac{2}{3} \ 0] \cdot [\frac{4}{3} \ -1] = [\frac{-6}{39} \ -12 \ 3]\)

10. \([\frac{2}{3} \ 0 \ -1] \cdot [\frac{2}{3} \ 0 \ -1] = [\frac{-2}{-2} \ -5]\)

11. \([4 \ 0 \ 2] \cdot [\frac{1}{2}] = [\frac{4}{12} \ 0 \ 6]\)

12. \([\frac{1}{3}] \cdot [4 \ 0 \ 2] = [\frac{4}{12} \ 0 \ 6]\)

13. \([\frac{6}{5} \ -1] \cdot [\frac{0}{5} \ 0] = [\frac{-30}{15} \ 10]\)

14. \([\frac{-15}{-1} \ -9] = [\frac{6}{12} \ 11 \ -297 \ -75]\)

Use \(A = [\frac{1}{2} \ \frac{2}{3}]\), \(B = [\frac{2}{-1} \ \frac{1}{0}]\), and scalar \(c = 3\) to determine whether the following equations are true for the given matrices.

15. \(AC = CA\) yes

16. \(AB + C = BA + CA\) no

17. \((AB)c = c(AB)\) yes

18. \((A + CB) = B(A + C)\) no

**RENTALS** For Exercises 19-21, use the following information.

For their one-week vacation, the Montoyas can rent a 2-bedroom condominium for $1796, a 3-bedroom condominium for $2165, or a 4-bedroom condominium for $2538. The table shows the number of units in each of three complexes.

<table>
<thead>
<tr>
<th>Complex</th>
<th>2-Bedroom</th>
<th>3-Bedroom</th>
<th>4-Bedroom</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun Haven</td>
<td>36</td>
<td>24</td>
<td>22</td>
</tr>
<tr>
<td>Surfside</td>
<td>29</td>
<td>32</td>
<td>42</td>
</tr>
<tr>
<td>Seabreeze</td>
<td>18</td>
<td>22</td>
<td>18</td>
</tr>
</tbody>
</table>

19. Write a matrix that represents the number of each type of unit available at each complex and a matrix that represents the weekly charge for each type of unit.

20. If all of the units in the three complexes are rented for the week at the rates given the Montoyas, express the income of each of the three complexes as a matrix.

21. What is the total income of all three complexes for the week? $526,054
1. Find the Error Both A and B are 2 by 2 matrices. Maggie made the following derivation. Is this derivation valid? If not, what error did she make?

a. \((A + B)^2 = (A + B)(A + B)\)

b. \(= (A + B)A + (A + B)B\)

c. \(= AA + BA + AB + BB\)

d. \(= A^2 + BA + AB + B^2\)

e. \(= A^2 + AB + AB + B^2\)

f. \(= A^2 + 2AB + B^2\)

From step d to step e, Maggie commuted A and B; but for matrices, \(AB\) does not typically equal \(BA\).

2. Exam Scores Mr. Farey recorded the exam scores of his students in a 20 by 3 matrix. Each row listed the scores of a different student. The first exam scores were listed in the first column, and the second exam scores were listed in the second column. The final exam scores were listed in the third column. Mr. Farey needed to create a 20 by 1 matrix that contained the weighted scores of each student. The first two exams account for 25% of the weighted score, and the final exam counted 50%. To make the matrix of weighted scores, what matrix can Mr. Farey multiply his 20 by 3 matrix by on the right?

\[
\begin{bmatrix}
0.25 & 0.25 & 0.50
\end{bmatrix}
\]

3. Special Matrices Mandy has a 3 by 3 matrix \(M\). She notices that for any 3 by 3 matrix \(X\), \(MX = X\). What must \(M\) be?

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

4. Powers Thad just learned about matrix multiplication. He began to wonder what happens when you take powers of a matrix. He computed the first few powers of the matrix \(M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\) and noticed a pattern. What is \(M^2\)?

\[
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\]

5. Cost Comparisons For Exercises 5 and 6, use the following information.

Barbara and Lance need to buy pens, pencils, and erasers. They make a 2 by 3 matrix that represents the numbers of each item they would like to purchase.

\[
\begin{bmatrix}
10 & 15 & 3 \\
5 & 20 & 5
\end{bmatrix}
\]

6. Compute \(MP\).

\[
\begin{bmatrix}
36.55 & 35.2 \\
31 & 31.75
\end{bmatrix}
\]

7. What do the entries in \(MP\) mean?

The first row represents the total cost of Barbara’s items at stores 1 and 2, respectively, and the second row represents the total cost of Lance’s items at stores 1 and 2, respectively.

8. Properties of Matrices

Computing with matrices is different from computing with real numbers. Stated below are some properties of the real number system. Are these also true for matrices? In the problems on this page, you will investigate this question.

a. \((A + B)C = AC + BC\)

b. \((AB)C = A(BC)\)

c. \(a(A + B) = aA + aB\)

d. \(aBC = (ab)C\)

For all real numbers \(a\) and \(b\), \(ab = 0\) if and only if \(a = 0\) or \(b = 0\).

Multiplication is commutative. For all real numbers \(a\) and \(b\), \(ab = ba\).

Multiplication is associative. For all real numbers \(a\), \(b\), and \(c\), \((abc) = (ab)c\).

Use the matrices \(A\), \(B\), and \(C\) for the problems. Write whether each statement is true. Assume that a 2-by-2 matrix is the 0 matrix if and only if all of its elements are zero.

\[
A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}
\]

1. \(AB = 0\) no
2. \(AC = 0\) no
3. \(BC = 0\) yes
4. \(AB = BA\) no
5. \(AC = CA\) no
6. \(BC = CB\) no

9. \(BAC = (BA)C\) yes

Both products equal \(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\).

10. Write a statement summarizing your findings about the properties of matrix multiplication.

Based on these examples, matrix multiplication is associative, but not commutative. Two matrices may have a product of zero even if neither of the factors equals zero.
4-4 Lesson Reading Guide
Transformations with Matrices

Get Ready for the Lesson
Read the introduction to Lesson 4-4 in your textbook.
Describe how you can change the orientation of a figure without changing its size or shape.
Flip (or reflect) the figure over a line.

Read the Lesson
1. Write the vertex matrix for the quadrilateral ABCD shown in the graph at the right.
   \[
   \begin{bmatrix}
   -4 & 2 & 1 & -2 \\
   1 & 3 & -4 & -3 \\
   \end{bmatrix}
   \]

2. Write the vertex matrix that represents the position of the quadrilateral A'B'C'D' that results when quadrilateral ABCD is translated 3 units to the right and 2 units downward.
   \[
   \begin{bmatrix}
   -1 & 5 & 4 & 1 \\
   -1 & 1 & -6 & -5 \\
   \end{bmatrix}
   \]

3. Describe the transformation that corresponds to each of the following matrices.
   a. \[
   \begin{bmatrix}
   -1 & 0 \\
   0 & -1 \\
   \end{bmatrix}
   \]
      counterclockwise rotation about the origin of 180°

   b. \[
   \begin{bmatrix}
   -4 & -4 & -4 \\
   -4 & -4 & -4 \\
   \end{bmatrix}
   \]
      translation 4 units down and 3 units to the right

   c. \[
   \begin{bmatrix}
   -1 & 0 \\
   0 & 1 \\
   \end{bmatrix}
   \]
      reflection over the y-axis

   d. \[
   \begin{bmatrix}
   0 & 1 \\
   1 & 0 \\
   \end{bmatrix}
   \]
      reflection over the line \( y = x \)

Remember What You Learned
3. Describe a way to remember which of the reflection matrices corresponds to reflection over the \( x \)-axis.
   Sample answer: The only elements used in the reflection matrices are 0, 1, and \(-1\). For such a \( 2 \times 2 \) matrix \( M \) to have the property that \( M \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \), the elements in the top row must be 1 and 0 (in that order), and elements in the bottom row must be 0 and \(-1\) (in that order).
4-4 Study Guide and Intervention (continued)
Transformations with Matrices

Reflections and Rotations

<table>
<thead>
<tr>
<th>Reflection Matrices</th>
<th>For a reflection over the:</th>
<th>axes</th>
<th>y-axis</th>
<th>line $y = x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>multiply the vertex matrix on the left by:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rotation Matrices</td>
<td>For a counterclockwise rotation about the origin:</td>
<td>$90^\circ$</td>
<td>$180^\circ$</td>
<td>$270^\circ$</td>
</tr>
<tr>
<td>multiply the vertex matrix on the left by:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example 1: Find the coordinates of the vertices of the image of $\triangle ABC$ with $A(3, 5), B(2, 4),$ and $C(1, -1)$ after a reflection over the line $y = x$.

Write the ordered pairs as a vertex matrix. Then multiply the vertex matrix by the reflection matrix for $y = x$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 \\ 5 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ -5 & 4 & -1 \end{pmatrix}$$

The coordinates of the vertices of $A'B'C'$ are $A'(5, 3), B'(4, -2),$ and $C'(-1, 1)$.

Exercises

1. The coordinates of the vertices of quadrilateral $ABCD$ are $A(2, 1), B(1, 3), C(2, 2),$ and $D(2, 1)$. What are the coordinates of the vertices of the image $A'B'C'D'$ after a reflection over the $y$-axis? $A'(2, 1), B'(1, 3), C'(-2, 2),$ and $D'(-2, -1)$.

2. Triangle $DEF$ with vertices $D(2, 5), E(1, 4),$ and $F(0, 1)$ is rotated $90^\circ$ counterclockwise about the origin.
   a. Write the coordinates of the triangle in a vertex matrix.
      $$\begin{pmatrix} -2 & 1 & 0 \\ 5 & 4 & -1 \end{pmatrix}$$
   b. Write the rotation matrix for this situation.
      $$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
   c. Find the coordinates of the vertices of $\triangle D'E'F'$. $D'(-5, -2), E'(-4, 1), F'(1, 0)$
   d. Graph $\triangle DEF$ and $\triangle D'E'F'$.

For Exercises 3-13, use the following information.

$\triangle ABC$ with vertices $A(2, 3), B(0, 4),$ and $C(-3, -3)$ is translated $3$ units right and $1$ unit down.

1. Write the translation matrix.
   $$\begin{pmatrix} 3 & 3 & 3 \\ -1 & -1 & -1 \end{pmatrix}$$

2. Find the coordinates of $\triangle A'B'C'. A'(5, 2), B'(3, 3), C'(0, -4)$

3. Graph the preimage and the image.

For Exercises 4-6, use the following information.

The vertices of $\triangle RST$ are $R(-3, 1), S(2, -1),$ and $T(1, 3)$. The triangle is dilated so that its perimeter is twice the original perimeter.

4. Write the coordinates of $\triangle RST$ in a vertex matrix.

5. Find the coordinates of the image $\triangle R'S'T'$.

6. Graph $\triangle RST$ and $\triangle R'S'T'$.

For Exercises 7-10, use the following information.

The vertices of $\triangle DEF$ are $D(4, 0), E(0, -1),$ and $F(2, 3)$. The triangle is reflected over the $x$-axis.

7. Write the coordinates of $\triangle DEF$ in a vertex matrix.

8. Write the reflection matrix for this situation.

9. Find the coordinates of $\triangle D'E'F'$. $D'(4, 0), E'(-0, 1), F'(2, -3)$

10. Graph $\triangle DEF$ and $\triangle D'E'F'$.

For Exercises 11-14, use the following information.

$\triangle XYZ$ with vertices $X(-1, -3), Y(-4, 1),$ and $Z(-2, 5)$ is rotated $180^\circ$ counterclockwise about the origin.

11. Write the coordinates of the triangle in a vertex matrix.

12. Write the rotation matrix for this situation.

13. Find the coordinates of $\triangle X'Y'Z'. X'(-1, 3), Y'(4, -1), Z'(2, -5)$

14. Graph the preimage and the image.
4-4
Practice
Transformations with Matrices

For Exercises 1-4, use the following information.

Quadrilateral WXYZ is translated 1 unit right and 3 units up. The vertices of WXYZ are (3, 2), (2, 4), (4, 1), and (2, 3), respectively.

1. Write the translation matrix.

\[
\begin{bmatrix}
0 & 0 & 1 & 3 \\
0 & 1 & 0 & 3 \\
\end{bmatrix}
\]

2. Find the coordinates of W’X’Y’Z’.

W’(4, 3), X’(5, 5), Y’(6, 4), Z’(5, 2)

3. Graph the translation and the image.

4. What is the vertex matrix for the image?

\[
\begin{bmatrix}
1 & 0 & 3 & 3 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 3 \\
\end{bmatrix}
\]

For Exercises 5-10, use the following information.

Quadrilateral ABCD is reflected over the x-axis to form the vertices A’(3, 2), B’(4, 1), C’(3, 5), and D’(1, 4). The vertices of ABCD are (2, 1), (1, 3), (3, 5), and (4, 1), respectively.

5. What is the vertex matrix for the image?

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

6. Find the coordinates of A’B’C’D’.

A’(2, 1), B’(3, 1), C’(5, 3), D’(4, 5)

7. Graph the quadrilateral and the image.

8. Write the reflection matrix for this situation.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

9. Find the coordinates of the vertices of the image.

A’(2, 1), B’(3, 1), C’(5, 3), D’(4, 5)

10. Graph A’B’C’D’ and A B C D.

11. ARCHITECTURE

Using architectural design software, the Bradleys plot their kitchen plans on a grid with each unit representing 1 foot. They place the corners of the island at (3, 2), (2, 3), (1, 1), and (0, 0), respectively.

a. Graph the island and its coordinates.

b. If the Bradleys wish to move the island 1.5 feet to the right and 2 feet down, what will the new coordinates of its corners be?

C’(3.5, 0.5), D’(2, -2), B’(3.5, 1), A’(2, -1)

12. BUSINESS

The design of a business logo calls for locating the vertices of a triangle at (1.5, 5), (4.1, 1), and (1, 0) on a grid. If the design changes require rotating the triangle 90º counterclockwise, what will the new coordinates of its corners be?

C’(5.5, 1.5), B’(2.5, -4.1), A’(-4.5, -1.5)
Computer Graphics Using Matrix Transformations

Computer animation creates a sensation of movement by slowly changing the position of pixels on an image on a two-dimensional video screen. By selecting the center of the screen as the origin, each pixel can be located using ordered pairs. A series of delayed matrix transformations (translation, pause, transformation, pause) applied to the coordinates of the point (pixel) produces the desired animation effect.

Two computer programmers, Nate and Daniel, are writing a computer animation program using only the two matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Nate argues that it does not make a difference which matrix is applied first. Daniel disagrees.

1. What transformation does each matrix represent?
   Matrix $A$ represents a reflection about the $x$-axis, and matrix $B$ represents a $90^\circ$ counterclockwise rotation.

2. Find $AB$ and $BA$. Determine whether Nate or Daniel is correct.
   
   $AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; Nate is correct because order does matter.

3. A third programmer joins Nate and Daniel. She convinces them to expand their transformation capabilities by including a reflection about the $y$-axis using $C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
   a. Determine a transformation using only $B$ and $C$ that is equivalent to the transformation $A$.
   
   $B^2 \cdot C = CA$
   
   b. Write an expression for a $180^\circ$ counterclockwise rotation in terms of the reflections $A$ and $C$.
   
   $B^2 = AC$ or $B^2 = CA$
   
   c. Is it possible to express a $270^\circ$ counterclockwise rotation in terms of the reflections $A$ and $C$? Explain why or why not.
   
   No; $B^3 = ABC$

Exercises

Example 1

Find the coordinates of the image of $\triangle ABC$ with vertices $A(0, 0), B(6, 0)$, and $C(3, 4)$ after a counterclockwise rotation of $30^\circ$ about the origin.


The coordinates of the image of $\triangle ABC$ are $A'(\sqrt{3}, 3), B'(5.2, 3)$, and $C'(0.6, 5.0)$.

Example 2

Find the coordinates of the image of $\triangle ABC$ with vertices $A(0, 0), B(6, 0)$, and $C(3, 4)$ after two rotations of $45^\circ$ counterclockwise about the origin.

In order to rotate the image twice, store the vertex matrix of the first image.

Keystrokes:


The vertices of $\triangle A'B'C'$ are $A'(0, 0), B'(0, 6)$, and $C'(-3, 4)$.

Exercises

Quadrilateral $ABCD$ has vertices $A(0, 0), B(4, 0), C(4, 6)$, and $D(0, 6)$. Find the coordinates of the vertices of the image after each counterclockwise rotation. Round to the nearest tenth.

1. $45^\circ$
   
   $A'(0, 0), B'(2.8, 2.8), C'(-1.4, 7.1), D'(-4.2, 4.2)$
   
2. $30^\circ$
   
   $A'(0, 0), B'(3.5, 2), C'(-3.5, 7.2), D'(-6.4, 2.2)$
   
3. $60^\circ$
   
   $A'(0, 0), B'(2.3, 5.2), C'(-3.2, 6.5), D'(-5.2, 3)$
Lesson Reading Guide

Determinants

Get Ready for the Lesson

Read the introduction to Lesson 4-5 in your textbook.

In this lesson, you will learn how to find the area of a triangle if you know the coordinates of its vertices using determinants. Describe a method you already know for finding the area of the Bermuda Triangle. Sample answer: Use the map. Choose any side of the triangle as the base, and measure this side with a ruler. Multiply this length by the scale factor for the map. Next, draw a segment from the opposite vertex perpendicular to the base. Measure this segment, and multiply its length by the scale for the map. Finally, find the area by using the formula $A = \frac{1}{2}bh$.

Read the Lesson

1. Indicate whether each of the following statements is true or false.

   a. Every matrix has a determinant. \textit{false}
   
   b. If both rows of a $2 \times 2$ matrix are identical, the determinant of the matrix will be 0. \textit{true}
   
   c. Every element of a $3 \times 3$ matrix has a minor. \textit{true}
   
   d. In order to evaluate a third-order determinant by expansion by minors it is necessary to find the minor of every element of the matrix. \textit{false}
   
   e. If you evaluate a third-order determinant by expansion about the second row, the position signs you will use are $- + -$. \textit{true}

2. Suppose that triangle $RST$ has vertices $R(-2, 5)$, $S(4, 1)$, and $T(0, 6)$.

   a. Write a determinant that could be used in finding the area of triangle $RST$.

      $\begin{vmatrix}
      -2 & 5 & 1 \\
      4 & 1 & 1 \\
      0 & 6 & 1 
      \end{vmatrix}$

   b. Explain how you would use the determinant you wrote in part a to find the area of the triangle. Sample answer: Evaluate the determinant and multiply the result by $\frac{1}{2}$. Then take the absolute value to make sure the final answer is positive.

Remember What You Learned

3. A good way to remember a complicated procedure is to break it down into steps. Write a list of steps for evaluating a third-order determinant using expansion by minors.

   Sample answer: 1. Choose a row of the matrix. 2. Find the position signs for the row you have chosen. 3. Find the minor of each element in that row. 4. Multiply each element by its position sign and by its minor. 5. Add the results.
Determinants of $3 \times 3$ Matrices

**Third-Order Determinants**

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

**Area of a Triangle**

The area of a triangle having vertices $(a, b, c)$ and $(d, e, f)$ is $|A|$, where

$$|A| = \frac{1}{2} \begin{vmatrix} a & b & c \\ d & e & f \\ 1 & 1 & 1 \end{vmatrix}$$

**Example**

Evaluate $\begin{vmatrix} 4 & 5 & -2 \\ 2 & -3 & 6 \\ -1 & 3 & 2 \end{vmatrix}$.

$$\begin{vmatrix} 4 & 5 & -2 \\ 2 & -3 & 6 \\ -1 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 5 & -2 \\ 2 & -3 & 6 \\ 1 & 3 & 2 \end{vmatrix}$$

Third-order determinant.

$$= 4(18 - 0) - 5(6 - 0) - 1(9)$$

$$= 72 - 30 + 9$$

$$= 51$$

**Exercises**

Evaluate each determinant.

1. $\begin{vmatrix} 3 & 0 & -2 \\ 2 & 5 & 3 \\ -1 & 4 & 6 \end{vmatrix}$

2. $\begin{vmatrix} 4 & 1 & 0 \\ 2 & -2 & 5 \\ -1 & 3 & 2 \end{vmatrix}$

3. $\begin{vmatrix} 6 & 1 & 4 \\ 2 & 3 & 0 \\ -1 & 3 & 2 \end{vmatrix}$

4. $\begin{vmatrix} 5 & -2 & 3 \\ 2 & 4 & -3 \\ 6 & 1 & -4 \end{vmatrix}$

5. $\begin{vmatrix} 6 & 1 & -4 \\ -2 & 2 & -1 \\ -1 & 3 & 2 \end{vmatrix}$

6. $\begin{vmatrix} 5 & -4 & 1 \\ -1 & 6 & -3 \\ 1 & 1 & 2 \end{vmatrix}$

7. Find the area of a triangle with vertices $(2, -3), (4, 7),$ and $(-5, 5)$.

44.5 square units
Answers (Lesson 4-5)

1. **Find the value of each determinant.**

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>3</td>
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<td>4</td>
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<td>6</td>
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<tr>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

Sample answer: $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$

2. **Evaluate each determinant using expansion by minors.**

<table>
<thead>
<tr>
<th>x</th>
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<tbody>
<tr>
<td>1</td>
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<tr>
<td>7</td>
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</tr>
</tbody>
</table>

Sample answer: $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$

3. **Evaluate each determinant using diagonals.**

<table>
<thead>
<tr>
<th>x</th>
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<tbody>
<tr>
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<td>7</td>
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<td>9</td>
</tr>
</tbody>
</table>

Sample answer: $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$

4. **ARROWS.** For Exercise 5 and 6, use the following information: vertices of triangle are $(0, 0)$, $(1, 0)$, and $(1, 1)$.

Evaluate the determinant that gives the area of this triangle.

Sample answer: $\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1/2$

5. **HALF-UNIT TRIANGLES.** For a school project, students had to decorate a pool. Ronald had studied determinants, so he wanted to make triangles with areas of one half square unit. Because the pool was rectangular, he had learned that this was essentially the same as finding the coordinates of the vertices of the triangle.

Given an example of such a triangle $\triangle ABC$ with $A(0, 0)$, $B(1, 0)$, and $C(0, 1)$.

Sample answer: $[0, 0, 1, 1] \
\{-1, 0, 1, 0\} \
\{0, 1, 0, 1\} = 1/2$

6. **LAND MANAGEMENT.** For a fish and wildlife management organization using a GIS (geographic information system) to store and analyze data for the parcels of land it manages, the coordinates of the vertices of a parcel of land are given as \(-2, -3\), \(-1, 4\), and \(5, 0\). If the coordinates of the vertices of a second parcel of land are \(-3, -2\), \(-2, 5\), and \(4, 0\), how many acres is the parcel? 133 acres.
Matrix Transpose and Determinants

In Lesson 4-1, you learned how to represent information in matrices. A matrix contains elements of the form $a_{ij}$, where $i$ is the row number of the element and $j$ is the column number of the element.

Consider the following matrix.

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$$

In this matrix, $a_{11} = 2, a_{12} = -1, a_{21} = 3,$ and $a_{22} = 4$.

The matrix transpose can be found by switching the elements around. Element $a_{ij}$ becomes element $a_{ji}$. So, the matrix transpose of $A$, denoted by $A^T$, is:

$$A^T = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

Calculate the determinant of $A$ and $A^T$.

$$\text{det}(A) = 2(4) - 3(-1) = 11$$
$$\text{det}(A^T) = 2(4) - (-1)(3) = 11$$

1. Find each matrix transpose.

   a. $B = \begin{bmatrix} -1 \\ 2 \\ 5 \\ 6 \end{bmatrix}$
   b. $C = \begin{bmatrix} 1 & 0 \\ -3 & 4 \end{bmatrix}$
   c. $D = \begin{bmatrix} 2 & -3 & -1 \\ 1 & 3 & -2 \end{bmatrix}$

   $$B^T = \begin{bmatrix} -1 \\ 2 \\ 5 \\ 6 \end{bmatrix}$$
   $$C^T = \begin{bmatrix} 1 & -3 & 0 \\ 4 & 3 & -1 \end{bmatrix}$$
   $$D^T = \begin{bmatrix} 2 & -3 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

2. Find the determinants of the original matrices and the transposes from Exercise 1.

   a. $\text{det}(B) = -16$; $\text{det}(B^T) = -16$; $\text{det}(C) = 7$; $\text{det}(C^T) = 7$;
   b. $\text{det}(D) = -18$; $\text{det}(D^T) = -18$

3. What do you notice about the determinants? Make a conjecture about the determinant of a matrix and the determinant of its transpose.

   They are the same. The determinant of a matrix is the same as the determinant of its transpose.

### Spreadsheet Activity

#### Cramer’s Rule

You have learned to solve systems of linear equations by using matrix equations and the inverse matrix. Another way to solve systems is to use Cramer’s Rule. Study the spreadsheet below to discover Cramer’s Rule.

To use the spreadsheet to solve a system of equations, write each equation in the form below:

$$ax + by = c$$

The values for the system $6x + 3y = -12$ and $5x + y = -8$ are shown. In the spreadsheet, the values of $a, b,$ and $c$ for the first equation are entered in cells $A1, B1,$ and $C1$, respectively. The values of $a, b,$ and $c$ for the second equation are entered in cells $A2, B2,$ and $C2$, respectively. The values in cells $B10$ and $B11$ represent the solution for the system.

### Exercises

1. Study the formula in cell A4. Write a matrix whose determinant is found using this formula.

   $$\begin{bmatrix} A1 \\ B2 \end{bmatrix}$$

2. Write matrices whose determinants are found using the formulas in cells $A6$ and $A8$.

   $$\begin{bmatrix} A1 & B1 \\ C2 & B2 \end{bmatrix}$$

3. Explain how the values of $x$ and $y$ are found using Cramer’s rule.

   $$x = \frac{(C1 * B2) - (A1 * C2)}{(A1 * B2) - (A2 * B1)}$$
   $$y = \frac{(C1 * A2) - (A1 * C2)}{(A1 * B2) - (A2 * B1)}$$

Use the spreadsheet to solve each system of equations.

4. $6x + 3y = -12$
   $5x + y = 8$

   $\begin{bmatrix} (4, -12) \\ (2, 8) \end{bmatrix}$

5. $5x - 3y = 19$
   $7x + 2y = 8$

   $\begin{bmatrix} (1, 6, 0.6) \\ (2, 3) \end{bmatrix}$

6. $6x - 3y = 11$
   $5x + 9y = 15$

   $\begin{bmatrix} (1, 6, 0.6) \\ (2, 3) \end{bmatrix}$

7. $3x + 1.6y = 0.44$
   $0.4x + 2.5y = 0.66$

   $\begin{bmatrix} (0.4, 0.2) \\ (2, 3) \end{bmatrix}$
Some students have trouble remembering how to arrange the determinants that are used in Cramer's Rule. Suppose that you are asked to solve the following system of equations by Cramer's Rule.

\[
\begin{align*}
3x + 2y &= 7 \\
2x - 3y &= 22
\end{align*}
\]

Without actually evaluating any determinants, indicate which of the following ratios of determinants gives the correct value for \(x\).

A. \(\frac{5}{1025}\)  B. \(\frac{5}{225}\)  C. \(\frac{10}{225}\)

Evaluate each determinant.

\[
\begin{align*}
D &= 3 \cdot 2 - 2 \cdot (-3) \\
&= 6 + 6 \\
&= 12
\end{align*}
\]

\[
\begin{align*}
D_x &= 3 \cdot 2 - 2 \cdot (-3) \\
&= 6 + 6 \\
&= 12
\end{align*}
\]

\[
\begin{align*}
D_y &= 3 \cdot 2 - 2 \cdot (-3) \\
&= 6 + 6 \\
&= 12
\end{align*}
\]

The solution is \(x = \frac{7}{12}\) and \(y = \frac{7}{12}\).

Use Cramer's Rule to solve each system of equations.

1. \(3x - 2y = 7\)
2. \(x - 4y = 17\)
3. \(2x - y = -2\)
4. \(2x - y = -1\)
5. \(4x + 2y = 1\)
6. \(6x - 3y = -3\)
7. \(2x + 7y = 16\)
8. \(2x - 3y = -2\)
9. \(x + 3y = 2\)
10. \(6x - 9y = -1\)
11. \(3x - 12y = -14\)
12. \(8x + 2y = 3\)

\[
\begin{align*}
D &= 5 \cdot 9 - 3 \cdot 4 \\
&= 45 - 12 \\
&= 33
\end{align*}
\]

\[
\begin{align*}
D_x &= 5 \cdot 9 - 3 \cdot 4 \\
&= 45 - 12 \\
&= 33
\end{align*}
\]

\[
\begin{align*}
D_y &= 5 \cdot 9 - 3 \cdot 4 \\
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\]

\[
\begin{align*}
D_y &= 5 \cdot 9 - 3 \cdot 4 \\
&= 45 - 12 \\
&= 33
\end{align*}
\]

The solution is \(\left(\frac{2}{3}, \frac{5}{9}\right)\), \(\left(-\frac{4}{3}, \frac{5}{6}\right)\), and \(\left(-\frac{1}{7}, \frac{11}{14}\right)\).
Study Guide and Intervention

Cramer’s Rule

Systems of Three Linear Equations

The solution of the system whose equations are

\[ \begin{align*}
ax + by + cz &= j \\
dx + ey + fz &= k \\
gx + hy + iz &= l
\end{align*} \]

is \((x, y, z)\) where

\[\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \neq 0\]

\[x = \frac{\begin{vmatrix} b & c & j \\ e & f & k \\ h & i & l \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}
\]

\[y = \frac{\begin{vmatrix} a & c & j \\ d & f & k \\ g & i & l \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}
\]

\[z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}
\]

Example

Use Cramer’s rule to solve the system of equations.

\[\begin{align*}
6x + 4y + z &= 5 \\
2x + 3y - 2z &= -2 \\
8x - 2y + 2z &= 10
\end{align*} \]

Use the coefficients and constants from the equations to form the determinants. Then evaluate each determinant.

\[\begin{vmatrix} 6 & 4 & 1 \\ 2 & 3 & -2 \\ 8 & -2 & 2 \end{vmatrix} = 6 \begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix} - 4 \begin{vmatrix} 2 & 1 \\ 8 & 2 \end{vmatrix} + 1 \begin{vmatrix} 2 & 3 \\ 8 & -2 \end{vmatrix} = 32 + 96 - 32 = 96 \]

\[\begin{vmatrix} 5 & 4 & 1 \\ 10 & -2 & 2 \\ 2 & 3 & -2 \end{vmatrix} = 5 \begin{vmatrix} -2 & 2 \\ 3 & -2 \end{vmatrix} - 4 \begin{vmatrix} 10 & 2 \\ 2 & 3 \end{vmatrix} + 1 \begin{vmatrix} 10 & -2 \\ 2 & 3 \end{vmatrix} = -20 - 44 + 24 = -60 \]

\[\begin{vmatrix} 5 & 4 & 1 \\ 10 & -2 & 2 \\ 2 & 3 & -2 \end{vmatrix} = 5 \begin{vmatrix} -2 & 2 \\ 3 & -2 \end{vmatrix} - 4 \begin{vmatrix} 10 & 2 \\ 2 & 3 \end{vmatrix} + 1 \begin{vmatrix} 10 & -2 \\ 2 & 3 \end{vmatrix} = -20 - 44 + 24 = -60 \]

\[\frac{96}{6} = 16 \]

\[\frac{-60}{6} = -10 \]

\[\frac{-32}{8} = -4 \]

The solution is \((x, y, z) = (5, 1, 3/3)\).

Exercise

Use Cramer’s rule to solve each system of equations.

1. \[\begin{align*}
x - 2y + 3z &= 6 \\
2x - y - z &= -3 \\
x + y + z &= 6
\end{align*} \] \((1, 2, 3)\)

2. \[\begin{align*}
x + y - 2z &= -2 \\
alx - 2y - 5z &= 7 \\
x + y + z &= 1
\end{align*} \] \((3, -5, 3)\)

3. \[\begin{align*}
x - 3y + 3z &= 1 \\
2x + 2y - z &= -8 \\
4x + 7y + 2z &= 11
\end{align*} \] \((-2, 1, 6)\)

4. \[\begin{align*}
x + y + 2z &= 1 \\
x + 3y + 2z &= -3 \\
x + y + 5z &= 0
\end{align*} \] \((-3, 3, 1)\)

5. \[\begin{align*}
3x + y + 2z &= 7 \\
2x - y - 3z &= -24 \\
10x + 3y - 2z &= -2
\end{align*} \] \((-3, 8, -2)\)

6. \[\begin{align*}
x + y + 4z &= 9 \\
x + y - 2z &= -13 \\
x + y - 7z &= 0
\end{align*} \] \((8, 5, 2)\)

Use Cramer’s Rule to solve each system of equations.

7. \[\begin{align*}
x + y - z &= 7 \\
x + 2y - 3z &= 2 \\
x + 3y - 4z &= 11
\end{align*} \] \((3, -5, 3)\)

8. \[\begin{align*}
x + 2y - 3z &= 2 \\
x + 2y - 3z &= 1 \\
x + 2y - 3z &= 3
\end{align*} \] \((-3, 3, 1)\)

9. \[\begin{align*}
x + y - z &= 7 \\
x + 2y - 3z &= 1 \\
x + 3y - 4z &= 0
\end{align*} \] \((-3, 8, -2)\)

10. \[\begin{align*}
x + y - z &= 7 \\
x + 2y - 3z &= 1 \\
x + 3y - 4z &= 0
\end{align*} \] \((-3, 8, -2)\)

11. \[\begin{align*}
x + y - z &= 7 \\
x + 2y - 3z &= 1 \\
x + 3y - 4z &= 0
\end{align*} \] \((-3, 8, -2)\)

12. \[\begin{align*}
x + y - z &= 7 \\
x + 2y - 3z &= 1 \\
x + 3y - 4z &= 0
\end{align*} \] \((-3, 8, -2)\)

13. \[\begin{align*}
x + y - z &= 7 \\
x + 2y - 3z &= 1 \\
x + 3y - 4z &= 0
\end{align*} \] \((-3, 8, -2)\)

14. \[\begin{align*}
x + y - z &= 7 \\
x + 2y - 3z &= 1 \\
x + 3y - 4z &= 0
\end{align*} \] \((-3, 8, -2)\)

15. \[\begin{align*}
x + y - z &= 7 \\
x + 2y - 3z &= 1 \\
x + 3y - 4z &= 0
\end{align*} \] \((-3, 8, -2)\)

16. \[\begin{align*}
x + y - z &= 7 \\
x + 2y - 3z &= 1 \\
x + 3y - 4z &= 0
\end{align*} \] \((-3, 8, -2)\)

17. GEOMETRY The two sides of an angle are contained in the lines whose equations are

\[\begin{align*}
x + y &= 4 \\
x - 3y &= -5
\end{align*} \]

Find the coordinates of the vertex of the angle. \((2, -1)\)

Use Cramer’s Rule to solve each system of equations.

18. \[\begin{align*}
x + y + z &= 2 \\
x + 2y + 3z &= 3 \\
x + 2y - 3z &= -3
\end{align*} \] \((2, 3, 4)\)

19. \[\begin{align*}
x + y - z &= 5 \\
x + 2y - 3z &= 12 \\
x + 3y - 4z &= 11
\end{align*} \] \((-2, 1, 6)\)

20. \[\begin{align*}
x + y - z &= 5 \\
x + 2y - 3z &= 12 \\
x + 3y - 4z &= 11
\end{align*} \] \((-2, 1, 6)\)

21. \[\begin{align*}
x + y - z &= 5 \\
x + 2y - 3z &= 12 \\
x + 3y - 4z &= 11
\end{align*} \] \((-2, 1, 6)\)
### Word Problem Practice

#### Cramer’s Rule

**1. USING CRAMER’S RULE** Lucy is solving the following system of linear equations using Cramer’s Rule.

\[
\begin{align*}
2x + 3y &= 5 \\
x + y &= 2
\end{align*}
\]

Write the three determinants she will have to compute.

\[
\begin{array}{ccc}
2 & 3 & 5 \\
1 & 1 & 2 \\
1 & 2 & 1 \\
\end{array}
\]

#### IMPULCATIONS OF CRAMER’S RULE

Cramer’s Rule gives the solutions of systems of linear equations in terms of their coefficients. The formula involves addition, subtraction, multiplication, and division of those coefficients. Is it possible for an irrational number to be part of the solution of a system of linear equations whose coefficients are all rational numbers?

No, It is impossible.

#### SHOPPING

Sheets cost $18.59 each and pillowcases cost $7.24 at Carol’s Linens. If Agatha buys \( x \) sheets and \( y \) pillowcases at Carol’s Linens, she’ll spend $210.75. On the other hand, at Save-n-Sleep, sheets cost $15.79 and pillowcases cost $8.19. If Agatha buys \( x \) sheets and \( y \) pillowcases at Save-n-Sleep, she’ll spend $191.25. Use Cramer’s Rule to determine how many sheets and pillow cases Agatha wants to buy.

9 sheets and 6 pillow cases

#### BRICKS

Linus owns three different types of brick that differ only in length. If he lines up 2 short, 1 medium, and 2 long bricks, the total length will be 45 inches. If he lines up 1 short, 2 medium, and 3 long bricks, the total length will be 59 inches. If he lines up 2 short, 1 medium, and 1 long brick, the total length will be 53 inches. Use Cramer’s Rule to determine how long the different types of brick are.

6.5, 9, and 11.5 in.

#### PROMOTIONS

For Exercises 5–7, use the following information.

A local zoo was trying to increase attendance by offering $22 for every child that came. However, the zoo insisted that there be at least 1 adult for every 8 children. A school decided to take advantage of the situation by sending 1 adult for every 8 children. Let \( c \) be the number of children and let \( a \) be the number of adults. Admission for adults was \( d \) dollars. The total cost of admission for everyone was $13.50.

5. Write a system of equations that describes the situation.

\[8a + 2c = 0 \quad da - 2c = 13.50\]

6. Is it possible that \( d = 16 \)? Explain in terms of Cramer’s Rule.

No, because Cramer’s Rule would then involve division by zero.

7. If adults were charged $20.50 for admission, how many adults and children went? Use Cramer’s Rule to solve.

3 adults and 24 children
**Identity and Inverse Matrices**

**Get Ready for the Lesson**

Read the introduction to Lesson 4-7 in your textbook. Refer to the code table given in the introduction to this lesson. Suppose that you receive a message coded by this system as follows:

\[
16 \quad 12 \quad 51 \quad 19 \quad 5 \quad 109 \quad 25 \quad 19 \quad 89 \quad 5 \quad 14 \quad 4.
\]

Decode the message. *Please be my friend.*

**Read the Lesson**

1. Indicate whether each of the following statements is true or false.
   
   a. Every element of an identity matrix is 1. **false**
   
   b. There is a \(3 \times 2\) identity matrix. **false**
   
   c. Two matrices are inverses of each other if their product is the identity matrix. **true**
   
   d. If \(M\) is a matrix, \(M^{-1}\) represents the reciprocal of \(M\). **false**
   
   e. No \(3 \times 2\) matrix has an inverse. **true**
   
   f. Every square matrix has an inverse. **false**
   
   g. If the two columns of a \(2 \times 2\) matrix are identical, the matrix does not have an inverse. **true**

2. Explain how to find the inverse of a \(2 \times 2\) matrix. Do not use any special mathematical symbols in your explanation.

   **Sample answer:** First find the determinant of the matrix. If it is zero, then the matrix has no inverse. If the determinant is not zero, form a new matrix as follows. Interchange the top left and bottom right elements. Change the signs but not the positions of the other two elements. Multiply the resulting matrix by the reciprocal of the determinant of the original matrix. The resulting matrix is the inverse of the original matrix.

**Remember What You Learned**

3. One way to remember something is to explain it to another person. Suppose that you are studying with a classmate who is having trouble remembering how to find the inverse of a \(2 \times 2\) matrix. He remembers how to move elements and change signs in the matrix, but thinks that he should multiply by the determinant of the original matrix. How can you help him remember that he must multiply by the reciprocal of this determinant?

   **Sample answer:** If the determinant of the matrix is 0, its reciprocal is undefined. This agrees with the fact that if the determinant of a matrix is 0, the matrix does not have an inverse.
Identity and Inverse Matrices

The identity matrix for matrix multiplication is a square matrix with 1s for every element of the main diagonal and zeros elsewhere. If an $n \times n$ matrix $A$ has an inverse $A^{-1}$, then $A \cdot A^{-1} = A^{-1} \cdot A = I$.

Example: Determine whether $X = \begin{bmatrix} 7 & 4 \\ 16 & 6 \end{bmatrix}$ and $Y = \begin{bmatrix} 3 & -2 \\ -5 & \frac{7}{2} \end{bmatrix}$ are inverse matrices.

Find $X \cdot Y$.

$X \cdot Y = \begin{bmatrix} 7 & 4 \\ 16 & 6 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -5 & \frac{7}{2} \end{bmatrix} = \begin{bmatrix} 21 - 20 - 14 + 14 \\ 10 - 30 - 20 + 21 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Find $Y \cdot X$.

$Y \cdot X = \begin{bmatrix} 3 & -2 \\ -5 & \frac{7}{2} \end{bmatrix} \begin{bmatrix} 7 & 4 \\ 16 & 6 \end{bmatrix} = \begin{bmatrix} 21 - 20 - 33 + 12 \\ -35 + 35 - 20 + 21 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Since $X \cdot Y = Y \cdot X = I$, $X$ and $Y$ are inverse matrices.

Exercises

Determine whether each pair of matrices are inverses.

1. Yes, Yes, No
2. Yes, Yes, No
3. Yes, Yes, No
4. Yes, Yes, No
5. Yes, Yes, No
6. Yes, Yes, No

Find the inverse of each matrix, if it exists.

1. $\begin{bmatrix} 24 & 12 \\ 8 & 4 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
3. $\begin{bmatrix} -10 & -10 \\ 30 & 10 \end{bmatrix}$

4. $\begin{bmatrix} 6 & 5 \\ 10 & 8 \end{bmatrix}$
5. $\begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}$
6. $\begin{bmatrix} 8 & 4 \\ 10 & 4 \end{bmatrix}$

If $ad - bc = 0$, the matrix does not have an inverse.
Lesson 4-7

Skills Practice

Identity and Inverse Matrices

Determine whether each pair of matrices are inverses.

1. \( X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \) \( Y = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \) yes

2. \( P = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \) \( Q = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \) yes

3. \( M = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \) \( N = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \) no

4. \( A = \begin{bmatrix} -2 & 5 \\ 1 & 2 \end{bmatrix} \) \( B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \) yes

5. \( V = \begin{bmatrix} 0 & 7 \\ 0 & 0 \end{bmatrix} \) \( W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) yes

6. \( X = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \) \( Y = \begin{bmatrix} -1 & 2 \\ 1 & 6 \end{bmatrix} \) yes

7. \( G = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} \) \( H = \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \) yes

8. \( D = \begin{bmatrix} -4 & -5 \\ 4 & -4 \end{bmatrix} \) \( E = \begin{bmatrix} -0.125 & -0.125 \\ 0.125 & -0.125 \end{bmatrix} \) no

Find the inverse of each matrix, if it exists.

9. \( \begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix} \) \( \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \) yes

10. \( \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \) \( \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \) yes

11. \( \begin{bmatrix} 9 & 6 \\ 2 & 0 \end{bmatrix} \) no inverse exists

12. \( \begin{bmatrix} 2 & -4 \\ 6 & 0 \end{bmatrix} \) \( \begin{bmatrix} 1 & -2 \\ -6 & 4 \end{bmatrix} \) yes

13. \( \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \) \( \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \) no inverse exists

14. \( \begin{bmatrix} 3 & 6 \\ -1 & 2 \end{bmatrix} \) \( \begin{bmatrix} 3 & 6 \\ -1 & 2 \end{bmatrix} \) no inverse exists

15. \( \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \) \( \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \) no inverse exists

16. \( \begin{bmatrix} -4 & 5 \\ 1 & 1 \end{bmatrix} \) \( \begin{bmatrix} 2 & -5 \\ 1 & 4 \end{bmatrix} \) no inverse exists

17. \( \begin{bmatrix} 0 & -7 \\ 7 & 0 \end{bmatrix} \) \( \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \) no inverse exists

18. \( \begin{bmatrix} 10 & 8 \\ -8 & 0 \end{bmatrix} \) \( \begin{bmatrix} 10 & 8 \\ -8 & 0 \end{bmatrix} \) no inverse exists

19. \( \begin{bmatrix} 10 & 10 \\ -8 & -8 \end{bmatrix} \) \( \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \) \( \begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix} \) \( \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \) \( \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \) no inverse exists

20. \( \begin{bmatrix} 6 & 10 \\ -8 & -8 \end{bmatrix} \) \( \begin{bmatrix} 10 & 8 \\ -8 & 0 \end{bmatrix} \) \( \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \) \( \begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix} \) \( \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \) \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) no inverse exists

Practice

Identity and Inverse Matrices

Determine whether each pair of matrices are inverses.

1. \( M = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \) \( N = \begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix} \) no

2. \( X = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} \) \( Y = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \) yes

3. \( A = \begin{bmatrix} -3 & 1 \\ -4 & 5 \end{bmatrix} \) \( B = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \) yes

4. \( P = \begin{bmatrix} 6 & 2 \\ -2 & 3 \end{bmatrix} \) \( Q = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} \) yes

Determine whether each statement is true or false.

5. All square matrices have multiplicative inverses. false

6. All square matrices have multiplicative identities. true

Find the inverse of each matrix, if it exists.

7. \( \begin{bmatrix} 4 & 3 \\ 1 & 8 \end{bmatrix} \) \( \begin{bmatrix} -3 & -5 \\ 8 & 4 \end{bmatrix} \) no inverse exists

8. \( \begin{bmatrix} 2 & 0 \\ 5 & 0 \end{bmatrix} \) \( \begin{bmatrix} 1 & 5 \\ 1 & 3 \end{bmatrix} \) no inverse exists

9. \( \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix} \) \( \begin{bmatrix} 7 & -3 \\ 4 & -1 \end{bmatrix} \) no inverse exists

10. \( \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix} \) \( \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix} \) yes

GEOMETRY

For Exercises 13–16, use the figure at the right.

13. Write the vertex matrix \( A \) for the rectangle.

14. Use matrix multiplication to find \( BA \) if \( B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) and \( A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \).

15. Graph the vertices of the transformed triangle on the previous graph. Describe the transformation. dilation by a scale factor of \( \frac{2}{3} \)

16. Make a conjecture about what transformation \( B^{-1} \) describes on a coordinate plane. dilation by a scale factor of \( \frac{2}{3} \)

17. CODES

Use the alphabet table below and the inverse of coding matrix \( C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \) to decode this message:

18. \( \begin{bmatrix} 10 & 10 \\ -8 & 0 \end{bmatrix} \) \( \begin{bmatrix} 10 & 8 \\ -8 & 0 \end{bmatrix} \) no inverse exists

19. \( \begin{bmatrix} 10 & 10 \\ -8 & 8 \end{bmatrix} \) \( \begin{bmatrix} 10 & 8 \\ -8 & 8 \end{bmatrix} \) no inverse exists

20. \( \begin{bmatrix} 10 & 10 \\ -8 & 8 \end{bmatrix} \) \( \begin{bmatrix} 10 & 8 \\ -8 & 8 \end{bmatrix} \) no inverse exists

Answer keys and explanations are not included in this document.
4-7 Word Problem Practice
Identity and Inverse Matrices

1. ROTATIONS Suppose \( R \) represents a counterclockwise rotation about the origin by an angle of 45°. For what values of \( n \) is \( R^n \) equal to the inverse of \( R \)?

2. SPECIAL MATRICES Norman only likes working with matrices whose determinant is 1. If \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is such a matrix, what is its inverse?

\[ \begin{bmatrix} d - b \\ -c & a \end{bmatrix} \]

3. CRYPTOGRAPHY A friend sends you a secret message that was coded using the coding matrix \( C = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \) and the alphabet table.

<table>
<thead>
<tr>
<th>CODE</th>
<th>( A )</th>
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<th>( E )</th>
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The message is 567 [354 4620] 388. What is the decoded message?

HELP

4. SELF-INVERSES Philip notices that any matrix with ones and negative ones on the diagonal and zeroes everywhere else has the property that it is its own inverse. Give an example of a 2 by 2 matrix that is its own inverse but has at least 1 nonzero number off the diagonal.

\[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

5. What is the determinant of \( G \)?

The determinant of \( G \) is 0.

6. Does the inverse of \( G \) exist? Explain. The inverse of the matrix does not exist because \( ad - bc = 0 \).

7. Determine a matrix operation that could be used to transform \( G \) into its Additive Identity matrix.

\( G^2 \)

Enrichment

Permutation Matrices

A permutation matrix is a square matrix in which each row and each column has one entry that is 1. All the other entries are 0. Find the inverse of a permutation matrix interchanging the rows and columns. For example, row 1 is interchanged with column 1, row 2 is interchanged with column 2.

\[ P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \]

\[ P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \]

Write this matrix. The two matrices are the same.

Solve each problem.

1. There is just one 2 \( \times \) 2 permutation matrix that is not also an identity matrix. Write this matrix.

\[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

The two matrices are the same.

2. Find the inverse of the matrix you wrote in Exercise 1. What do you notice?

\[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

3. Show that the two matrices in Exercises 1 and 2 are inverses.

\[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

4. Write the inverse of this matrix.

\[ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ B^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

5. Use \( B^{-1} \) from problem 4. Verify that \( B \) and \( B^{-1} \) are inverses.

\[ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \]

6. Permutation matrices can be used to write and decipher codes. To see how this is done, use the message matrix \( M \) and matrix \( B \) from problem 4. Find matrix \( C \) so that \( C \) equals the product \( MB \). Use the rules below.

- 0 times a letter = 0
- 1 times a letter = the same letter
- 0 plus a letter = the same letter
- 1 plus a letter = the same letter

\[ \begin{bmatrix} H \ E \ S \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \]

Multiply \( M \) by \( B \) encodes the message. To decipher, multiply by \( B^{-1} \).

\[ \begin{bmatrix} H \ E \ S \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \]

MULTIPLYING BY \( B \) ENCODES THE MESSAGE. TO DECRYPT, MULTIPLY BY \( B^{-1} \).
Explain how to use the matrix equation you wrote above to solve the system. Use as many mathematical symbols as you can. Do not actually solve the system.

Sample answer: Find the inverse of the \( 2 \times 2 \) matrix of coefficients. Multiply this inverse by the \( 2 \times 1 \) matrix of constants, with the \( 2 \times 1 \) matrix on the left. The product will be a \( 2 \times 1 \) matrix. The number in the first row will be the value of \( x \), and the number in the second row will be the value of \( y \).

2. Write a system of equations that corresponds to the following matrix equation.
\[
\begin{bmatrix}
3 & 2 & -4 \\
2 & -1 & 0 \\
0 & 5 & 6
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
-2 \\
6 \\
-4
\end{bmatrix}
\]

\[
3x + 2y - 4z = -2 \\
2x - y = 6 \\
5y + 6z = -4
\]

Remember What You Learned

3. Some students have trouble remembering how to set up a matrix equation to solve a system of linear equations. What is an easy way to remember the order in which to write the three matrices that make up the equation?

Sample answer: Just remember “CVC” for “coefficients, variables, constants.” The variable matrix is on the left side of the equals sign, just as the variables are in the system of linear equations. The constant matrix is on the right side of the equals sign, just as the constants are in the system of linear equations.
Answers (Lesson 4-8)

Using Matrices to Solve Systems of Equations

Write a matrix equation for each system of equations.

1. \(2x + y = 5\)
2. \(3x + 2y = 6\)
3. \(x + y = 0\)
4. \(2x - y = 1\)
5. \(3x + 2y = 12\)
6. \(4x + y = 5\)
7. \(5x - y = 20\)
8. \(8x - y = -5\)

Solve each matrix equation or system of equations by using inverse matrices.

9. \([-1, -2] [-3, 1] = [5, 3]\)
10. \([-1, -2, 0] [-3, 1, 0] = [5, 3, 0]\)
11. \([-1, 0, 1] [-2, 0, 1] = [5, 3, 0]\)
12. \([-1, 0, 1] [-2, 0, 1] = [5, 3, 0]\)
13. \([-1, 0, 1] [-2, 0, 1] = [5, 3, 0]\)
14. \([-1, 0, 1] [-2, 0, 1] = [5, 3, 0]\)
15. \([-1, 0, 1] [-2, 0, 1] = [5, 3, 0]\)
16. \([-1, 0, 1] [-2, 0, 1] = [5, 3, 0]\)

Chapter 4
4-8 Practice

Using Matrices to Solve Systems of Equations

Write a matrix equation for each system of equations.

1. \[ \begin{bmatrix} 3 \end{bmatrix} x + \begin{bmatrix} 2 \end{bmatrix} y = \begin{bmatrix} 9 \end{bmatrix} \]
   \[ \begin{bmatrix} 5 \end{bmatrix} x - \begin{bmatrix} 3 \end{bmatrix} y = \begin{bmatrix} -13 \end{bmatrix} \]

2. \[ \begin{bmatrix} 2 \end{bmatrix} x - \begin{bmatrix} 2 \end{bmatrix} y = \begin{bmatrix} -2 \end{bmatrix} \]
   \[ \begin{bmatrix} 3 \end{bmatrix} x + \begin{bmatrix} 3 \end{bmatrix} y = \begin{bmatrix} 10 \end{bmatrix} \]

3. \[ \begin{bmatrix} 2 \end{bmatrix} a + \begin{bmatrix} 2 \end{bmatrix} b = \begin{bmatrix} 0 \end{bmatrix} \]
   \[ \begin{bmatrix} 3 \end{bmatrix} a + \begin{bmatrix} 2 \end{bmatrix} b = \begin{bmatrix} 2 \end{bmatrix} \]

4. \[ \begin{bmatrix} 4 \end{bmatrix} r + \begin{bmatrix} 5 \end{bmatrix} s = \begin{bmatrix} 10 \end{bmatrix} \]
   \[ \begin{bmatrix} 1 \end{bmatrix} r - \begin{bmatrix} 2 \end{bmatrix} s = \begin{bmatrix} -2 \end{bmatrix} \]

5. \[ \begin{bmatrix} 3 \end{bmatrix} x + \begin{bmatrix} 2 \end{bmatrix} y + \begin{bmatrix} 5 \end{bmatrix} z = \begin{bmatrix} 3 \end{bmatrix} \]
   \[ \begin{bmatrix} 1 \end{bmatrix} x - \begin{bmatrix} 1 \end{bmatrix} y + \begin{bmatrix} 5 \end{bmatrix} z = \begin{bmatrix} 2 \end{bmatrix} \]
   \[ \begin{bmatrix} -2 \end{bmatrix} x + \begin{bmatrix} 2 \end{bmatrix} y - \begin{bmatrix} 2 \end{bmatrix} z = \begin{bmatrix} -6 \end{bmatrix} \]

Solve each matrix equation or system of equations by using inverse matrices.

7. \[ \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} 3 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} -4 \end{bmatrix} \]

8. \[ \begin{bmatrix} -2 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 3 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} -7 \end{bmatrix} \begin{bmatrix} 5 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \]

9. \[ \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} 4 \end{bmatrix} \begin{bmatrix} 3 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} a \end{bmatrix} = \begin{bmatrix} 12 \end{bmatrix} \begin{bmatrix} 3 \end{bmatrix} \begin{bmatrix} -5 \end{bmatrix} \]

10. \[ \begin{bmatrix} 5 \end{bmatrix} \begin{bmatrix} 6 \end{bmatrix} \begin{bmatrix} 4 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 16 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 7 \end{bmatrix} \]

11. \[ \begin{bmatrix} -4 \end{bmatrix} \begin{bmatrix} 7 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} r \end{bmatrix} = \begin{bmatrix} -17 \end{bmatrix} \begin{bmatrix} 4 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \]

12. \[ \begin{bmatrix} 12 \end{bmatrix} \begin{bmatrix} 8 \end{bmatrix} \begin{bmatrix} 6 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} 3 \end{bmatrix} \begin{bmatrix} m \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 3 \end{bmatrix} \]

13. \[ \begin{bmatrix} 2 \end{bmatrix} x + \begin{bmatrix} 3 \end{bmatrix} y = \begin{bmatrix} 5 \end{bmatrix} \]
   \[ \begin{bmatrix} 4 \end{bmatrix} x - \begin{bmatrix} 2 \end{bmatrix} y = \begin{bmatrix} 1 \end{bmatrix} \]

14. \[ \begin{bmatrix} 8 \end{bmatrix} d + \begin{bmatrix} 2 \end{bmatrix} f = \begin{bmatrix} 13 \end{bmatrix} \]
   \[ \begin{bmatrix} -6 \end{bmatrix} d + \begin{bmatrix} 5 \end{bmatrix} f = \begin{bmatrix} -45 \end{bmatrix} \]

15. \[ \begin{bmatrix} 5 \end{bmatrix} m + \begin{batrix} 9 \end{bmatrix} n = \begin{bmatrix} 19 \end{bmatrix} \]
   \[ \begin{bmatrix} 2 \end{bmatrix} m - \begin{batrix} n \end{bmatrix} = \begin{bmatrix} -20 \end{bmatrix} \]

17. \[ \begin{bmatrix} 3 \end{bmatrix} x + \begin{bmatrix} 4 \end{bmatrix} y + \begin{bmatrix} 8 \end{bmatrix} z = \begin{bmatrix} 18 \end{bmatrix} \]
   \[ \begin{bmatrix} 2 \end{bmatrix} x + \begin{bmatrix} 3 \end{bmatrix} y + \begin{bmatrix} 5 \end{bmatrix} z = \begin{bmatrix} 13 \end{bmatrix} \]

18. \[ \begin{bmatrix} 2 \end{bmatrix} a + \begin{bmatrix} 3 \end{bmatrix} b + \begin{bmatrix} 5 \end{bmatrix} c = \begin{bmatrix} 12 \end{bmatrix} \]
   \[ \begin{bmatrix} 4 \end{bmatrix} a + \begin{bmatrix} 3 \end{bmatrix} b + \begin{bmatrix} 5 \end{bmatrix} c = \begin{bmatrix} 20 \end{bmatrix} \]

19. \[ \begin{bmatrix} 6 \end{bmatrix} x + \begin{bmatrix} 7 \end{bmatrix} y + \begin{bmatrix} 8 \end{bmatrix} z = \begin{bmatrix} 27 \end{bmatrix} \]
   \[ \begin{bmatrix} 2 \end{bmatrix} x + \begin{bmatrix} 3 \end{bmatrix} y + \begin{bmatrix} 5 \end{bmatrix} z = \begin{bmatrix} 18 \end{bmatrix} \]

SALES For Exercises 5 and 6, use the following information.

The school film society is selling only granola bars and oranges to raise money at their movie review. They sell oranges for $1 and granola bars for $1.50. The person who bought all of the granola bars purchased for this sale? f = n

5. Suppose a person spent $d to buy n items. Write a system of linear equations that relate d and n to the number of oranges r and granola bars g that the person purchased.

\[ h + k = n \]
\[ h + 1.5 k = d \]

6. One recorded sale showed that 10 items were purchased for $13.00. How many oranges and granola bars were purchased for this sale?

4 granola bars and 6 oranges

Chapter 4

Answers

Lessons 4-8

Glencoe Algebra 2

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Chapter 4
Determining Political Popularity

Systems of equations have applications in branches of science including chemistry, ecology, and physics. They can also be used to describe situations involving social studies and politics.

Consider the two candidates for City Council, Jefferson Dailey and Robert Jackson. Support for a candidate is measured by a positive number less than 1 and opposition of a candidate by a negative number greater than \(-1\). For example, 0.75 indicates fairly high support, while \(-0.75\) means fairly high opposition. Simultaneous support for, or opposition to, both candidates is possible. Generally, however, if one candidate is popular and is supported while the other candidate is opposed, support of the popular candidate tends to decrease as support for the "underdog" rises. Let the change in support for Jefferson Dailey be denoted by \(\Delta J\) (delta J) and the change in support for Robert Jackson is denoted by \(\Delta R\) (delta R).

This situation is described by the system of equations:

\[
\begin{align*}
\Delta J &= -0.5J + 0.25R \\
\Delta R &= -0.25J - 0.5R
\end{align*}
\]

For example, if \(\Delta J = -0.2\) and \(\Delta R = 0.2\), then current support for Jefferson Dailey is decreasing at a rate of 20% while Robert Jackson's support is increasing at 20%.

Substituting the given values for \(\Delta J\) and \(\Delta R\) and solving the first equation for \(R\) yields:

\[R = \frac{0.5J - 0.2}{0.25}\]

Substituting this expression for \(R\) in the second equation and solving for \(J\) yields:

\[0.2 = 0.25J - 0.5 \left( \frac{0.5J - 0.2}{0.25} \right) \Rightarrow J = 0.267\]

Therefore, \(R = -0.267\).

If the election were held today, Jefferson Daily would win.

Solve the systems of equations for the following values of \(\Delta J\) and \(\Delta R\) to determine potential winners and losers.

1. \(\Delta J = -0.1, \Delta R = 0.2\)
   
   \[J = 0, R = -0.4. \text{ Jefferson Daily is "winning".}\]

2. \(\Delta J = 0.5, \Delta R = -0.1\)
   
   \((-1.2, -0.4). \text{ Neither is favored, but Robert Jackson is ahead.}\)